Spin-Electromagnetics:

Spin-Vector-Potential-Coupling-Induced Force, Correlation between Rashba SOC, GMR/TMR and Family of Hall effects

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Abstract

We establish Universal Mathematical Field Equations (UMFE). Applying UMFE, we re-derive Maxwell equations, which justifies UMFE and shows that the experiments-based Maxwell equations have their mathematical origin; and establish Classical-Spin-Electromagnetics (C-Spin-EM) including Spin-Lorentz-type force and Lagrangian-Lorentz-type force. C-Spin-EM is self-consistence, powerful and fruitful, at classical level, in the perspective of fundamental physics: (1) universally explains and correlates family of Hall effects, zero longitudinal Hall coefficient/resistivity, Extended Rashbe SOC, and GMR/TMR. (2) predicts that Spin-potential coupling directly induce force, which contributes to Aharonov–Bohm effect; (3) provides classical counterparts of Larmor-precession, Stark Effect, Landau–Lifshitz equation, Zeeman effect, and Aharonov–Casher effect; (4) propose that electric field induces spin precession. UMFE shows that mathematical identities lead to physical dualities including duality between Electromagnetics and C-Spin-EM. We postulate a duality between Lagrangian-Lorentz force and Hamiltonian.

Key words: spin-electromagnetics, spintronics, Rashba effect, anomalous Hall effect, spin Hall effect, topological insulator, GMR/TMR

Contents

1. Introduction

In the macroscopic world, Electromagnetics (EM) was established based on the experiments. In the microscopic world, phenomena, such as the family of Hall effects, topological insulators, Rashba SOC, and GMR/TMR draw many activities. Let's briefly review those topics.

EM: There is a heuristic phenomenon: a stationary electric particle Q_e (abbreviated e-particle) induces only a static vector electric field. However, an observer moving relative to the same e-particle observes not only an electric field but also a magnetic field. The fundamental differences between static electric field and magnetic field are the following:

(1) The sources, "e-particle Q_e" vs. "e-current $J_e = Q_e\mathbf{v}$";

(2) The way the fields induced by source, "$\nabla \cdot \mathbf{E}$" vs. "$\nabla\times\mathbf{B}$";

(3) The nature of the fields, "vector field **E**" vs. "axial vector field **B**";

(4) Effects of the fields on a test e-particles, "$q_e\mathbf{E}$" vs. "$q_e\mathbf{v}\times\mathbf{B}$".

A physics student may ask questions: Why the motion of e-particles induces magnetic fields? Does the generation of magnetic fields relate with the Coulomb's law? A teacher's answer is that magnetism is the combination of electric field with Special Relativity and does not relate with the Coulomb's law. We argue that the teacher's answer is not sufficient to explain the above four fundamental differences.

Historically, Ampere law and Faraday law were established based on series experiments. Now we ask further questions: Is the experimental result of the generation of magnetic fields *inevitable*? The answer is yes. Thus we argue that: (1) the magnetic fields must be induced by the combination of the Coulomb's law and the velocity of e-particles; and, further, (2) Maxwell equations, except the Coulomb's law, must be derivable mathematically.

Q-Spin-EM: There are several fundamental quantum phenomena, such as Rashba SOC, Spin Hall effect, Anomalous Hall effect, and GMR/TMR, relating with spin; and zero longitudinal Hall coefficient/resistivity. We have questions: do those phenomena have classical counterparts and/or origins; if the answer is yes, is there a universal explanation and/or mechanism for those classical counterparts; what are further predictions of it for testing?

Spin-electromagnetism has been developed in quantum regime (abbreviated Q-Spin-EM) [1]. The classical correspondence of spin of e-particles is its angular momentum around its axis, denoted as $\mathbf{S_c}$. We postulate that, at classical level, the spin of e-particles induces new fields, and field equations can be derived mathematically. Furthermore, we expect that the same mathematical approach derive both EM and Classical-Spin-EM (abbreviated C-Spin-EM) due to electric charge and spin respectively.

Motivation: To address above questions motivate me to establish a set of Universal Mathematical Field Equations (abbreviated **UMFE**), such that UMFE is equally applicable to derive EM and C-Spin-EM, thus, there is duality between both, so that many concepts and effects of the well-established EM can be transferred directly to C-Spin-EM. Most important, C-Spin-EM and EM are linked together clearly and closely.

EM is foundation of Electronics; my goal is that C-Spin-EM may serve as a classical basis of Spintronics.

The significances of UMFE are the following.

Firstly, for justifying UMFE, combining UMFE and the Coulomb's law re-derives EM *mathematically* regardless spin, and answer the above-mentioned questions related with EM.

Secondly, combining UMFE and the Coulomb's law derives *mathematically* C-Spin-EM related with spin of e-particles. C-Spin-EM is powerful and fruitful:

(1) is explicitly connected with EM;
(2) predicts spin-waves;
(3) predicts Spin-Lorentz-type force and Lagrangian-Lorentz-type force; the latter contains spin-potential coupling, which acts as a force;
(4) predicts a Landau–Lifshitz-type equation as a supplement of Spin-Lorentz-type force;
(5) Spin-Lorentz-type and Lagrangian-Lorentz-type force cause Dual-Hall Effect, Topological Insulator, Extended-Hall Effect, Temperature Dependence of Extended-Hall Effect, Lagrangian-Hall effect; and shows, at classical level, longitudinal Hall coefficient/resistivity is zero, which is closely related with GMR/TMR;
(6) proposes Extended-Rashba SOC;
(7) suggests several new effects, such as Spin-Aharonov–Bohm Effect, spin-Aharonov–Casher effect, Spin-Larmor Precession and Spin-Stark Effect.

Thirdly, UMFE provides the mathematic origin of dualities between different physic fields derived from it, such as duality between electricity and magnetism, and duality between EM and C-Spin-EM. Duality is a powerful tool to find intrinsic similarities between apparently different phenomena, and predict new effects. "It turns out that most of the important concepts and theories of physics can be unified and understood by their common attribute of duality" (Damian P Hampshire).

In Part 1, we derive UMFE related to uniform motion and to spin respectively. In part 2, we apply UMFE to physical fields of e-particles to derive EM. Then we mathematically establish systematically a theoretical framework, C-Spin-EM, for studying spin related classic phenomena.

Part 1: Universal Mathematical Field Equations and Duality

2. UMFE and Duality

To make descriptions clear in this paper, let's clarify two terms:

(1) Fundamental laws: Physic laws describing fields induced by stationary sources, such as the inverse-square law, are fundamental laws, and cannot be derived mathematically from any other physics law.
(2) Secondary laws: Physic laws describing axial vector fields induced by moving sources are secondary laws and should be derivable mathematically from fundamental laws.

2.1. General UMFE

We need to find a vector analysis identity that connecting divergences of either a vector or an axial vector, and curl of an induced axial vector. For this aim, the following mathematical identity is the most noteworthy,

$$\nabla\times(\mathbf{S}\times\mathbf{T}) = \mathbf{S}(\nabla\cdot\mathbf{T}) - \mathbf{T}(\nabla\cdot\mathbf{S}) + (\mathbf{T}\cdot\nabla)\mathbf{S} - (\mathbf{S}\cdot\nabla)\mathbf{T}, \tag{2.1}$$

which indicates that the combination of gradient and divergence of two arbitrary vectors induces inevitably an axial vector. One of two terms, $(\nabla\cdot\mathbf{T})$ and $(\nabla\cdot\mathbf{S})$, represents fundamental inverse-square laws. It is useful to write Eq. (2.1) in a different but equivalent form. By using another mathematical identity,

$$(\mathbf{T}\cdot\boldsymbol{\nabla})\mathbf{S}=\boldsymbol{\nabla}(\mathbf{S}\cdot\mathbf{T})-(\mathbf{S}\cdot\boldsymbol{\nabla})\mathbf{T}-\mathbf{S}\times(\boldsymbol{\nabla}\times\mathbf{T})-\mathbf{T}\times(\boldsymbol{\nabla}\times\mathbf{S}),$$

Eq. (2.1) can be rewritten as an identity,

$$\nabla\times(\mathbf{S}\times\mathbf{T})=\mathbf{S}(\nabla\cdot\mathbf{T})-\mathbf{T}(\nabla\cdot\mathbf{S})-\nabla(\mathbf{S}\cdot\mathbf{T})+2(\mathbf{T}\cdot\nabla)\mathbf{S}+\mathbf{S}\times(\nabla\times\mathbf{T})+\mathbf{T}\times(\nabla\times\mathbf{S}). \quad (2.2)$$

Eq. (2.1) and Eq. (2.2) are mathematical equivalent. When apply UMFE to describe physical fields, the "**S**" and "**T**" in Eq. (2.1) and Eq. (2.2) represent any physical quantity.

For the purpose of this article, we set the "**S**" being motion parameters, velocity $\mathbf{v}$ and classical spin $\mathbf{S}_{\mathbf{clasical}}$, (abbreviated $\mathbf{S}_{\mathbf{c}}$),

$$\left.\begin{array}{r}\mathbf{S}=\mathbf{v}\ \text{(vector)}\\ \mathbf{S}=\mathbf{S}_{\mathbf{c}}\ \text{(axial vector)}\end{array}\right\}. \quad (2.3)$$

Note: the physical quantities **S** represents are not limited to those listed in Eq. (2.3).

Substituting Eq. (2.3) into Eq. (2.1) and Eq. (2.2) respectively, we obtain,

$$\nabla\times(\mathbf{v}\times\mathbf{T})=\mathbf{v}(\nabla\cdot\mathbf{T})-\mathbf{T}(\nabla\cdot\mathbf{v})+(\mathbf{T}\cdot\nabla)\mathbf{v}-(\mathbf{v}\cdot\nabla)\mathbf{T}, \quad (2.4)$$

$$\nabla\times(\mathbf{v}\times\mathbf{T})=\mathbf{v}(\nabla\cdot\mathbf{T})-\mathbf{T}(\nabla\cdot\mathbf{v})-\nabla(\mathbf{v}\cdot\mathbf{T})+2(\mathbf{T}\cdot\nabla)\mathbf{v}+\mathbf{v}\times(\nabla\times\mathbf{T})+\mathbf{T}\times(\nabla\times\mathbf{v}). \quad (2.5)$$

$$\nabla\times(\mathbf{S}_{\mathbf{c}}\times\mathbf{T})=\mathbf{S}_{\mathbf{c}}(\nabla\cdot\mathbf{T})-\mathbf{T}(\nabla\cdot\mathbf{S}_{\mathbf{c}})+(\mathbf{T}\cdot\nabla)\mathbf{S}_{\mathbf{c}}-(\mathbf{S}_{\mathbf{c}}\cdot\nabla)\mathbf{T}, \quad (2.6)$$

$$\nabla\times(\mathbf{S}_{\mathbf{c}}\times\mathbf{T})=\mathbf{S}_{\mathbf{c}}(\nabla\cdot\mathbf{T})-\mathbf{T}(\nabla\cdot\mathbf{S}_{\mathbf{c}})-\nabla(\mathbf{S}_{\mathbf{c}}\cdot\mathbf{T})+2(\mathbf{T}\cdot\nabla)\mathbf{S}_{\mathbf{c}}+\mathbf{S}_{\mathbf{c}}\times(\nabla\times\mathbf{T})+$$
$$+\mathbf{T}\times(\nabla\times\mathbf{S}_{\mathbf{c}}). \quad (2.7)$$

2.2. Duality

"Duality is one of the most fruitful ideas in Mathematics, has constantly been generalized and has guided the development of Mathematics. Duality in mathematics is not a theorem, but a 'principle'. It has a simple origin, it is very powerful and useful. Fundamentally duality gives two different points of view of looking at the same object, which in principle are all dualities. In some cases, 'duality' and 'symmetry' means essentially the same thing" [2].

We suggest that the contrary is true. Namely, duality gives one point of view of looking at the different objects, which in principle are all dualities. We show that this point of view of duality is also "powerful, fruitful and guidance" in this article.

Eq. (2.1), Eq. (2.2), and Eq. (2.4) to Eq. (2.7) are mathematically equivalent and, thus, are dual to each other. For studying dualities between different fields that are cross products of combinations of the different "**S**" and "**T**", we argue:

"Mathematical identities lead to mathematical dualities that lead to physical dualities. Duality discloses the similarity of intrinsic nature of apparently different interactions".

2.2.1. Type-1 duality and Type-2 duality

The "**T**" in Eq. (2.4) to Eq. (2.7) may be either a vector field $\mathbf{T}_{\mathbf{vector}}$ (abbreviated $\mathbf{T}_{\mathbf{v}}$) or an axial vector field $\mathbf{T}_{\text{axial-vector}}$ (abbreviated $\mathbf{T}_{\mathbf{av}}$) or a field $\mathbf{T}_{\text{combination}}$ (abbreviated $\mathbf{T}_{\mathbf{c}}$) that is the combination of a vector and an axial vector $\mathbf{T}_{\mathbf{c}} = \mathbf{T}_{\mathbf{v}} + \mathbf{T}_{\mathbf{av}}$.

For convenient in studying dualities, with different "**T**", let's introduce different categories of axial vector fields:

First level axial vector field (abbreviated **FAF**): "**c**" is defined as the cross product of $\mathbf{a}$ and **b,**

$\mathbf{c} \equiv \mathbf{a}\times\mathbf{b}$, where both the "**a**" and "**b**" are *vector field*, denote "**c**" as FAF.

Second level axial vector field (abbreviated **SAF**): "**d**" is defined as the cross product of $\mathbf{e}$ and **f,**

$\mathbf{d} \equiv \mathbf{e}\times\mathbf{f}$, where the "**e**" is a *vector field* and "**f**" is a *first level axial vector field*, denote "**d**" as SAF.

Third level axial vector field (abbreviated **TAF**): "**n**" is defined as the cross product of $\mathbf{q}$ and **p,**

$\mathbf{n} \equiv \mathbf{q}\times\mathbf{p}$, where both the "**q**" and "**p**" are *first level axial vector fields*, denote "**n**" as TAF.

To summarize, let's assign different numbers to vector and axial vector: "1" to vector; "2" to first level axial vector; "3" to second lever axial vector, and so on. The lever of a cross product of two quantities is: # of first quantity plus the # of second quantity subtract 1, then the result number is the level of the cross product.

Now let's introduce two categories of dualities as following:

Type-1 duality: for an axial field $\mathbf{W}_{\text{before}} \equiv \mathbf{S}\times\mathbf{T}$, under transformation(s) of either $\mathbf{S}$ or $\mathbf{T}$ or both $\mathbf{S}$ and $\mathbf{T}$, the field $\mathbf{W}_{\text{before}}$ transfers to $\mathbf{W}_{\text{after}}$. Under two conditions: (1) $\mathbf{W}_{\text{after}}$ and $\mathbf{W}_{\text{before}}$ are same level axial vector field; (2) the field equations describing respectively $\mathbf{W}_{\text{after}}$ and $\mathbf{W}_{\text{before}}$ have either the same form or are mathematically equivalent; then there is a Type-1 duality between $\mathbf{W}_{\text{after}}$ and $\mathbf{W}_{\text{before}}$.

Type-2 duality: for an axial field $\mathbf{W}_{\text{before}} \equiv \mathbf{S}\times\mathbf{T}$, under transformation(s) of either $\mathbf{S}$ or $\mathbf{T}$ or both $\mathbf{S}$ and $\mathbf{T}$, the field $\mathbf{W}_{\text{before}}$ transfers to $\mathbf{W}_{\text{after}}$. Under two conditions: (1) $\mathbf{W}_{\text{after}}$ and $\mathbf{W}_{\text{before}}$ are different level axial vector fields; (2) the field equations describing respectively $\mathbf{W}_{\text{after}}$ and $\mathbf{W}_{\text{before}}$ have either the same form or are mathematically equivalent; then there is a Type-2 duality between $\mathbf{W}_{\text{after}}$ and $\mathbf{W}_{\text{before}}$.

Note: during the transformation of Type-2 duality, at least one term in equation will become zero. For keeping the same form of equations, we still keep the zero-term for the purpose of discussing duality. Then, in later calculation, ignore those zero-terms.

Specific examples of type-1 duality:

In the following examples, n is an integer and n = 1, 2, 3….

Example 1: Corresponding to different axial vector fields $\mathbf{T}_{\mathbf{avn}}$, the fields $\mathbf{v}\times\mathbf{T}_{\mathbf{avn}}$ are SAF. Dualities between those SAFs are type-1 duality, i.e., under transformation,

$$\mathbf{T}_{\mathbf{av1}} \leftrightarrow \cdots \leftrightarrow \mathbf{T}_{\mathbf{avn}}$$

there are conversions between the SAFs,

$$\mathbf{v}\times\mathbf{T}_{\mathbf{av1}} \leftrightarrow \cdots \leftrightarrow \mathbf{v}\times\mathbf{T}_{\mathbf{avn}}, (\mathbf{n} \neq \mathbf{1})$$

and between UMFE describing them.

Example 2: Corresponding to different axial vector fields $\mathbf{T}_{\mathbf{avn}}$, the fields $\mathbf{S}_{c}\times\mathbf{T}_{\mathbf{avn}}$ are TAF. Dualities between those TAFs are type-1 duality, i.e., under transformation,

$$\mathbf{T}_{\mathbf{av1}} \leftrightarrow \cdots \leftrightarrow \mathbf{T}_{\mathbf{avn}}$$

we have conversions between the TAFs,

$$\mathbf{S}_c\times\mathbf{T}_{\mathbf{av1}} \leftrightarrow \cdots \leftrightarrow \mathbf{S}_c\times\mathbf{T}_{\mathbf{avn}}, (\mathbf{n} \neq \mathbf{1})$$

and between UMFE describing them.

Specific examples of type-2 duality:

Example 1: Corresponding to a vector field $\mathbf{T}_{\mathbf{vn}}$ and an axial vector field $\mathbf{T}_{\mathbf{avn}}$, the fields $\mathbf{v}\times\mathbf{T}_{\mathbf{vn}}$ and fields $\mathbf{v}\times\mathbf{T}_{\mathbf{avn}}$ are FAF and SAF respectively.

Duality between $\mathbf{v}\times\mathbf{T}_{\mathbf{vn}}$ and $\mathbf{v}\times\mathbf{T}_{\mathbf{avn}}$ is type-2 duality.

Example 2: same as Example 1, but replace $\mathbf{v}$ with $\mathbf{S}_c$.

Example 3: Corresponding to a vector field $\mathbf{T}_{\mathbf{vn}}$, the field $\mathbf{v}\times\mathbf{T}_{\mathbf{vn}}$ and field $\mathbf{S}_c\times\mathbf{T}_{\mathbf{vn}}$ are FAF and SAF respectively. Duality between $\mathbf{v}\times\mathbf{T}_{\mathbf{vn}}$ and $\mathbf{S}_c\times\mathbf{T}_{\mathbf{vn}}$ is type-2 duality.

Example 4: same as Example 3, but replace $\mathbf{T}_{\mathbf{vn}}$ with $\mathbf{T}_{\mathbf{avn}}$.

2.2.2. Transferability between Dualities

The mathematical type-1 duality and type-2 duality can be transferred. We propose Transfer Rules:

(1) There are type-1 duality between A and B, and type-1 duality between C and D. If the duality between A and C is type-1, then the duality between B and D is type-1, and vice versa;

(2) There are type-2 duality between A and B, and type-2 duality between C and D. If the duality between A and C is type-1, and if B and D are the same lever axial fields, then the duality between B and D is type-1, and vice versa;

(3) There are type-1 (or type 2) duality between A and B; and C is mathematical equivalent to A. There is D that is mathematical equivalent to B, and is type-1 (or type 2) dual to C.

(4) There is type-1 (or type 2) duality between A and B; and C is mathematical equivalent to A. There is a type-1 (or type 2) dual of C, which is mathematical equivalent to B.

2.3. UMFE Related with Velocity of Charges

The basic concept is that the combination of the inverse-square laws and the motion of charges must induce axial vector fields. Fortunately, Eq. (2.1) and Eq. (2.2) link the motion of sources and the inverse-square laws. In this section we start with Eq. (2.4) to derive the Maxwell-type equations for the fields induced by the velocity of sources.

Note in Eq. (2.4), the velocity is spatially varying, e.g., $\mathbf{T}(\nabla\cdot\mathbf{v}) \neq 0$, $(\mathbf{T}\cdot\nabla)\mathbf{v} \neq 0$, and is instantaneous velocity at a given space point. Eq. (2.4) implies that a velocity and its spatial variations induce the axial vector $(\mathbf{v}\times\mathbf{T})$ field.

2.3.1. Ampere-Maxwell-type UMFE

Firstly, we derive the Ampere-Maxwell-type UMFE. Let's assume the $\mathbf{T}$ is an arbitrary vector field $\mathbf{G}$ and $\nabla\cdot\mathbf{G} \neq 0$. Since

$$-(\mathbf{v}\cdot\nabla)\mathbf{G} = -\left[\mathrm{v_x}\frac{\partial}{\partial \mathrm{x}} + \mathrm{v_y}\frac{\partial}{\partial \mathrm{y}} + \mathrm{v_z}\frac{\partial}{\partial \mathrm{z}}\right]\mathbf{G} = \frac{\partial \mathbf{G}}{\partial \mathrm{t}} - \frac{\mathrm{d}\mathbf{G}}{\mathrm{dt}}, \quad (2.8)$$

substituting Eq. (2.8) into Eq. (2.4), we obtain the Ampere-Maxwell-type UMFE

$$\nabla\times(\mathbf{v}\times\mathbf{G}) + \frac{\mathrm{d}\mathbf{G}}{\mathrm{dt}} = \mathbf{v}(\nabla\cdot\mathbf{G}) + \frac{\partial \mathbf{G}}{\partial \mathrm{t}} - \mathbf{G}(\nabla\cdot\mathbf{v}) + (\mathbf{G}\cdot\nabla)\mathbf{v}. \quad (2.9)$$

Defining a FAF $\mathbf{M}$,

$$\mathbf{M} \equiv \mathbf{v}\times\mathbf{G}. \tag{2.10}$$

For the axial vector field **M**, we have,

$$\nabla\cdot\mathbf{M} = 0. \tag{2.11}$$

Substituting Eq. (2.10) into Eq. (2.4) and Eq. (2.9) respectively, we obtain

$$\nabla\times\mathbf{M} = \mathbf{v}(\nabla\cdot\mathbf{G}) - \mathbf{G}(\nabla\cdot\mathbf{v}) + (\mathbf{G}\cdot\nabla)\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{G}. \tag{2.12}$$

$$\nabla\times\mathbf{M} + \frac{\mathrm{d}\mathbf{G}}{\mathrm{dt}} = \mathbf{v}(\nabla\cdot\mathbf{G}) + \frac{\partial\mathbf{G}}{\partial\mathrm{t}} - \mathbf{G}(\nabla\cdot\mathbf{v}) + (\mathbf{G}\cdot\nabla)\mathbf{v}. \tag{2.13}$$

All terms on the right hand side of Eq. (2.13) induce equally the axial vector field **M**. The interpretations for those terms are,

1) The term, $\mathbf{v}(\nabla\cdot\mathbf{G})$, plays the role of the "current";
2) The term, $\frac{\partial\mathbf{G}}{\partial\mathrm{t}}$, plays the role of the "displacement current";
3) The term, $\mathbf{G}(\nabla\cdot\mathbf{v})$, describes stretching of the **G** field due to source velocity compressibility;
4) The term, $(\mathbf{G}\cdot\nabla)\mathbf{v}$, describes the stretching or tilting of the **G** field due to the velocity gradients;
5) The terms, $\frac{\partial\mathbf{G}}{\partial\mathrm{t}}$ and $\mathbf{G}(\nabla\cdot\mathbf{v})$, have the same direction; while the terms, $\mathbf{v}(\nabla\cdot\mathbf{G})$ and $(\mathbf{G}\cdot\nabla)\mathbf{v}$, have the same direction.

The Ampere-Maxwell-type UMFE, Eq. (2.9), Eq. (2.12), Eq. (2.13), can be written respectively in the integral form as,

$$\oint(\mathbf{v}\times\mathbf{G})\cdot\mathrm{d}\mathbf{l} + \frac{\mathrm{d}}{\mathrm{dt}}\iint\mathbf{G}\cdot\mathrm{ds} = \iint\left[\mathbf{v}(\nabla\cdot\mathbf{G}) + \frac{\partial\mathbf{G}}{\partial\mathrm{t}} - \mathbf{G}(\nabla\cdot\mathbf{v}) + (\mathbf{G}\cdot\nabla)\mathbf{v}\right]\cdot\mathrm{ds}. \tag{2.14}$$

$$\oint\mathbf{M}\cdot\mathrm{d}\mathbf{l} = \int[\mathbf{v}(\nabla\cdot\mathbf{G}) - \mathbf{G}(\nabla\cdot\mathbf{v}) + (\mathbf{G}\cdot\nabla)\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{G}]\cdot\mathrm{ds}. \tag{2.15}$$

$$\oint\mathbf{M}\cdot\mathrm{d}\mathbf{l} + \frac{\mathrm{d}}{\mathrm{dt}}\iint\mathbf{G}\cdot\mathrm{ds} = \int\left[\mathbf{v}(\nabla\cdot\mathbf{G}) + \frac{\partial\mathbf{G}}{\partial\mathrm{t}} - \mathbf{G}(\nabla\cdot\mathbf{v}) + (\mathbf{G}\cdot\nabla)\mathbf{v}\right]\cdot\mathrm{ds}. \tag{2.16}$$

2.3.2. Faraday-type UMFE

For deriving the Faraday-type UMFE, taking the **T** field as a FAF **M** defined by Eq. (2.10), $\mathbf{T} = \mathbf{M}$. For the **M** field, we have,

$$-(\mathbf{v}\cdot\nabla)\mathbf{M} = -\left[\mathrm{v_x}\frac{\partial}{\partial\mathrm{x}} + \mathrm{v_y}\frac{\partial}{\partial\mathrm{y}} + \mathrm{v_z}\frac{\partial}{\partial\mathrm{z}}\right]\mathbf{M} = \frac{\partial\mathbf{M}}{\partial\mathrm{t}} - \frac{\mathrm{d}\mathbf{M}}{\mathrm{dt}}. \tag{2.17}$$

Substituting Eq. (2.17) into Eq. (2.4), we obtain Faraday-type UMFE,

$$\nabla\times(\mathbf{v}\times\mathbf{M}) + \frac{\mathbf{dM}}{\mathbf{dt}} = \mathbf{v}(\nabla\cdot\mathbf{M}) + \frac{\partial\mathbf{M}}{\partial\mathbf{t}} - \mathbf{M}(\nabla\cdot\mathbf{v}) + (\mathbf{M}\cdot\nabla)\mathbf{v}. \tag{2.18}$$

Let's define a SAF **N**,

$$\mathbf{N} \equiv -\mathbf{v}\times\mathbf{M}, \tag{2.19}$$

which satisfies,

$$\nabla\cdot\mathbf{N} = 0. \tag{2.20}$$

Substituting Eq. (2.19) into Eq. (2.18), we obtain Faraday-type UMFE,

$$\nabla\times\mathbf{N} - \frac{\mathbf{dM}}{\mathbf{dt}} = -\mathbf{v}(\nabla\cdot\mathbf{M}) - \frac{\partial\mathbf{M}}{\partial\mathbf{t}} + \mathbf{M}(\nabla\cdot\mathbf{v}) - (\mathbf{M}\cdot\nabla)\mathbf{v}. \tag{2.21}$$

The term $\mathbf{v}(\nabla\cdot\mathbf{M})$ is the source term. We still keep the source term in Eq. (2.21), because, this source term leaves a door open for a possible existence of a monopole physically, although, mathematically, the monopole of the **M** field does not exist.

The interpretations of those right hand side terms of Eq. (2.21) are,

1) The term, $\mathbf{v}(\nabla\cdot\mathbf{M})$, plays the role of the "current", which is mathematically zero;

2) The term, $\frac{\partial\mathbf{M}}{\partial t}$, describes the time change of the **M** fields as the source;

3) The term, $\mathbf{M}(\nabla\cdot\mathbf{v})$, describes stretching of the **M** field due to source velocity compressibility;

4) The term, $(\mathbf{M}\cdot\nabla)\mathbf{v}$, describes the stretching or tilting of the **M** field due to the velocity gradients;

5) The terms, $\frac{\partial\mathbf{M}}{\partial t}$ and $\mathbf{M}(\nabla\cdot\mathbf{v})$, have the same direction; while the terms, $\mathbf{v}(\nabla\cdot\mathbf{M})$ and $(\mathbf{M}\cdot\nabla)\mathbf{v}$, have the same direction.

Next taking $\nabla\cdot\mathbf{M}=0$, substituting it into Eq. (2.18) and Eq. (2.21) respectively, we obtain source-free Faraday-type UMFE,

$$\nabla\times(\mathbf{v}\times\mathbf{M})+\frac{\mathbf{dM}}{\mathbf{dt}}=\frac{\partial\mathbf{M}}{\partial\mathbf{t}}-\mathbf{M}(\nabla\cdot\mathbf{v})+(\mathbf{M}\cdot\nabla)\mathbf{v}, \tag{2.22}$$

$$\nabla\times\mathbf{N}-\frac{\mathbf{dM}}{\mathbf{dt}}=-\frac{\partial\mathbf{M}}{\partial\mathbf{t}}+\mathbf{M}(\nabla\cdot\mathbf{v})-(\mathbf{M}\cdot\nabla)\mathbf{v}. \tag{2.23}$$

The Faraday-type UMFE, Eq. (2.18) and Eq. (2.21), can be written respectively in the integral form as,

$$\oint(\mathbf{v}\times\mathbf{M})\cdot\mathrm{d}\mathbf{l}+\frac{\mathrm{d}}{\mathrm{dt}}\iint\mathbf{M}\cdot\mathrm{ds}=\iint\left[\mathbf{v}(\nabla\cdot\mathbf{M})+\frac{\partial\mathbf{M}}{\partial\mathbf{t}}-\mathbf{M}(\nabla\cdot\mathbf{v})+(\mathbf{M}\cdot\nabla)\mathbf{v}\right]\cdot\mathrm{ds}. \tag{2.24}$$

$$\oint\mathbf{N}\cdot\mathrm{d}\mathbf{l}-\frac{\mathrm{d}}{\mathrm{dt}}\iint\mathbf{M}\cdot\mathrm{ds}=-\iint\left[\mathbf{v}(\nabla\cdot\mathbf{M})+\frac{\partial\mathbf{M}}{\partial\mathbf{t}}-\mathbf{M}(\nabla\cdot\mathbf{v})+(\mathbf{M}\cdot\nabla)\mathbf{v}\right]\cdot\mathrm{ds}. \tag{2.25}$$

2.3.3. Type-2 Duality

The **M** field is a FAF, while the **N** field is a SAF. The **M** field and the **N** field are determined respectively by Eq. (2.16) and Eq. (2.25), which have the same form. Thus there is type-2 duality between the FAF **M** and the SAF **N**, which is pre-determined mathematically.

We have derived the basic UMFE for the fields induced by velocity of sources.

2.3.4. UMFE Related with Non-Spatially-Varying Velocity of Charges

For non-spatially-varying, we have

$$\nabla\cdot\mathbf{v}=(\mathbf{T}\cdot\nabla)\mathbf{v}=\nabla\times\mathbf{v}=0, \tag{2.26}$$

Substituting Eq. (2.26) into Eq. (2.13), we obtain Ampere-Maxwell-type UMFE for the **M** field,

$$\nabla\times\mathbf{M}+\frac{\mathrm{d}\mathbf{G}}{\mathrm{dt}}=\mathbf{v}(\nabla\cdot\mathbf{G})+\frac{\partial\mathbf{G}}{\partial t}. \tag{2.27}$$

Or in the integral forms,

$$\oint\mathbf{M}\cdot\mathrm{d}\mathbf{l}+\frac{\mathrm{d}}{\mathrm{dt}}\iint\mathbf{G}\cdot\mathrm{ds}=\iint\left[\mathbf{v}(\nabla\cdot\mathbf{G})+\frac{\partial\mathbf{G}}{\partial\mathbf{t}}\right]\cdot\mathrm{ds}. \tag{2.28}$$

Substituting Eq. (2.26) into Eq. (2.23), we obtain Faraday-type UMFE for the **N** field,

$$\nabla\times\mathbf{N}-\frac{\mathbf{dM}}{\mathbf{dt}}=-\frac{\partial\mathbf{M}}{\partial\mathbf{t}}. \tag{2.29}$$

Or in the integral forms,

$$\oint\mathbf{N}\cdot\mathrm{d}\mathbf{l}-\frac{\mathrm{d}}{\mathrm{dt}}\iint\mathbf{M}\cdot\mathrm{ds}=-\iint\frac{\partial\mathbf{M}}{\partial t}\cdot\mathrm{ds}. \tag{2.30}$$

2.4. UMFE Related with Spin

In this section, we derive UMFE governing fields induced by Spin of charges, where spin has no physical meaning, only represents a states of motion.

2.4.1. General UMFE

Eq. (2.7) implies that the spin of charges indeed induces inevitably an axial vector field, $(\mathbf{S_c}\times\mathbf{T})$. We start with Eq. (2.7). For a vector field **T**, the $(\mathbf{S_c}\times\mathbf{T})$ is a SAF; for an axial field **T**, the $(\mathbf{S_c}\times\mathbf{T})$ is a TAF.

Following the definitions, $\mathbf{M}\equiv\mathbf{v}\times\mathbf{G}$ and $\mathbf{N}\equiv-\mathbf{v}\times\mathbf{M}$, let's define a SAF **W** and a TAF **Z**:

$$\mathbf{W}\equiv\mathbf{S_c}\times\mathbf{G}, \tag{2.31}$$

$$\mathbf{Z}\equiv-\mathbf{S_c}\times\mathbf{M}. \tag{2.32}$$

The SAF **W** and the TAF **Z** satisfy respectively,

$$\nabla\cdot\mathbf{W}=0, \tag{2.33}$$

$$\nabla\cdot\mathbf{Z}=0. \tag{2.34}$$

Firstly, derive Ampere-type UMFE for the **W** field. Let $\mathbf{T}=\mathbf{G}$, and $\nabla\cdot\mathbf{G}\neq0$. Substituting Eq. (2.31) into Eq. (2.7), we obtain Ampere-type UMFE for the **W** field,

$$\nabla\times\mathbf{W}=\mathbf{S_c}(\nabla\cdot\mathbf{G})-\mathbf{G}(\nabla\cdot\mathbf{S_c})-\nabla(\mathbf{S_c}\cdot\mathbf{G})+2(\mathbf{G}\cdot\nabla)\mathbf{S_c}+\mathbf{S_c}\times(\nabla\times\mathbf{G})+$$
$$+\mathbf{G}\times(\nabla\times\mathbf{S_c}). \tag{2.35}$$

Secondly, derive Ampere-type UMFE for the **Z** field. Let $\mathbf{T}=\mathbf{M}$, where **M** is the FAF defined by Eq. (2.10). Substituting Eq. (2.32) into Eq. (2.7), we obtain Ampere-type UMFE for the **Z** field,

$$\nabla\times\mathbf{Z}=-\mathbf{S_c}(\nabla\cdot\mathbf{M})+\mathbf{M}(\nabla\cdot\mathbf{S_c})+\nabla(\mathbf{S_c}\cdot\mathbf{M})-2(\mathbf{M}\cdot\nabla)\mathbf{S_c}-\mathbf{S_c}\times(\nabla\times\mathbf{M})-$$
$$-\mathbf{M}\times(\nabla\times\mathbf{S_c}). \tag{2.36}$$

2.4.2. Type-2 Duality

The equations of the **W** field, Eq. (2.35), and the **Z** field, Eq. (2.36), are all derived from the same equations, Eq. (2.7), thus there is type-2 duality between the SAF **W** and the TAF **Z** fields.

2.4.3. Faraday-type UMFE

Substituting Eq. (2.13) into Eq. (2.36), we obtain the Faraday-type UMFE,

$$\nabla\times\mathbf{Z}=-\frac{\partial\mathbf{W}}{\partial t}-\mathbf{S_c}(\nabla\cdot\mathbf{M})+\mathbf{M}(\nabla\cdot\mathbf{S_c})+\nabla(\mathbf{S_c}\cdot\mathbf{M})-2(\mathbf{M}\cdot\nabla)\mathbf{S_c}-\mathbf{M}\times(\nabla\times\mathbf{S_c})$$
$$-\mathbf{S_c}\times\left\{-\frac{d\mathbf{G}}{dt}+\mathbf{v}(\nabla\cdot\mathbf{G})-\mathbf{G}(\nabla\cdot\mathbf{v})+(\mathbf{G}\cdot\nabla)\mathbf{v}\right\}+\frac{\partial\mathbf{S_c}}{\partial t}\times\mathbf{G}. \tag{2.37}$$

2.4.4. UMFE Related with Non-Spatially-Varying Spin of Charges

For non-spatially varying spin $\mathbf{S_c}$ and velocity **v**, we have

$$\nabla\cdot\mathbf{S_c}=(\mathbf{M}\cdot\nabla)\mathbf{S_c}=(\mathbf{G}\cdot\nabla)\mathbf{S_c}=\boldsymbol{\nabla}\times\mathbf{S_c}=\nabla\cdot\mathbf{v}=(\mathbf{G}\cdot\nabla)\mathbf{v}=0.$$

Substituting into Eq. (2.35) and Eq. (2.37) respectively, we obtain

$$\nabla\times\mathbf{W}=\mathbf{S_c}(\nabla\cdot\mathbf{G})-\nabla(\mathbf{S_c}\cdot\mathbf{G})+\mathbf{S_c}\times(\nabla\times\mathbf{G}). \tag{2.38}$$

$$\nabla\times\mathbf{Z}=-\frac{\partial\mathbf{W}}{\partial t}-\mathbf{S_c}(\nabla\cdot\mathbf{M})+\nabla(\mathbf{S_c}\cdot\mathbf{M})-\mathbf{S_c}\times\left\{-\frac{d\mathbf{G}}{dt}+\mathbf{v}(\nabla\cdot\mathbf{G})\right\}+\frac{\partial\mathbf{S_c}}{\partial t}\times\mathbf{G}. \tag{2.39}$$

2.5. Dualities between UMFEs Related with Velocity and Spin

UMFE describing fields induced by velocity and spin are derived from the same mathematical formulas, Eq. (2.1). Thus they are equivalent mathematically. Therefore the duality between those fields are determined by their definitions, FAF including: $\mathbf{M} \equiv \mathbf{v}\times\mathbf{G}$,

SAF including: $\mathbf{N} \equiv -\mathbf{v}\times\mathbf{M}$, $\mathbf{W} \equiv \mathbf{S_c}\times\mathbf{G}$,

TAF including: $\mathbf{Z} \equiv -\mathbf{S_c}\times\mathbf{M}$.

Therefore there is type-1 duality between SAF fields, i.e., **N** and **W** field. There are type-2 dualities between different level axial fields: (1) between FAF and SAF, i.e., **M** and **N** fields, **M** and **W** fields; (2) between FAF and TAF, i.e., **M** and **Z** fields; and (3) between SAF and TAF, i.e., **N** and **Z** fields, **W** and **Z** fields.

Part 2: Electromagnetics and Classical-Spin-Electromagnetics

3. Extended EM Derived from UMFE and Coulomb's Law

We re-derive mathematically Maxwell equations from UMFE and the Coulomb's law to show that UMFE is valid for physic fields, which leads us to apply UMFE to other physical fields, such as spin-induced fields.

3.1. Extended Faraday's Law

Starting from the Faraday-type UMFE, Eq. (2.28). Let the **M** field is a magnetic field **B,** $\mathbf{M} = \mathbf{B}$, Eq. (2.28) gives ,

$$\oint -(\mathbf{v}\times\mathbf{B}) \cdot \mathrm{d}\mathbf{l} - \frac{\mathrm{d}}{\mathrm{dt}}\iint \mathbf{B} \cdot \mathrm{ds} = -\iint \left[\mathbf{v}(\nabla \cdot \mathbf{B}) + \frac{\partial \mathbf{B}}{\partial \mathbf{t}} - \mathbf{B}(\nabla \cdot \mathbf{v}) + (\mathbf{B} \cdot \nabla)\mathbf{v}\right] \cdot \mathrm{ds}. \quad (3.1)$$

Let's show that Eq. (3.1) consists with Faraday's law.

The Faraday's law gives

$$\frac{\mathrm{d}}{\mathrm{dt}}\iint \mathbf{B} \cdot \mathrm{ds} = -\oint \mathbf{E}' \cdot \mathrm{dl} \quad (3.2)$$

where $\mathbf{E}'$ is the electric field at the circuit d**l** in a reference frame in which d**l** is at rest. The **B** is a magnetic field at the neighborhood of the circuit.

Applying Eq. (3.2), Eq. (3.1) becomes the extended Faraday law,

$$\oint (\mathbf{E}' - \mathbf{v}\times\mathbf{B}) \cdot dl = -\iint \left[\mathbf{v}(\nabla \cdot \mathbf{B}) + \frac{\partial \mathbf{B}}{\partial \mathbf{t}} - \mathbf{B}(\nabla \cdot \mathbf{v}) + (\mathbf{B} \cdot \nabla)\mathbf{v}\right] \cdot \mathrm{ds}, \quad (3.3)$$

where the **v** is the velocity of the circuit relative to a laboratory frame.

From Galilean invariance, one can define an electric field **E** in the laboratory frame,

$$\mathbf{E} = \mathbf{E}' - \mathbf{v}\times\mathbf{B}. \quad (3.4)$$

Applying Eq. (3.4), Eq. (3.3) gives the integral and differential forms of Extended Faraday's law,

$$\oint \mathbf{E} \cdot dl = -\iint \left[\mathbf{v}(\nabla \cdot \mathbf{B}) + \frac{\partial \mathbf{B}}{\partial \mathbf{t}} - \mathbf{B}(\nabla \cdot \mathbf{v}) + (\mathbf{B} \cdot \nabla)\mathbf{v}\right] \cdot \mathrm{ds}, \quad (3.5)$$

$$\nabla\times\mathbf{E} = -\mathbf{v}(\nabla \cdot \mathbf{B}) - \frac{\partial \mathbf{B}}{\partial t} + \mathbf{B}(\nabla \cdot \mathbf{v}) - (\mathbf{B} \cdot \nabla)\mathbf{v}. \quad (3.6)$$

Let's define a "current" generating induced electric field, denote it as

$$\mathbf{j_{v-E}} \equiv \mathbf{v}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{v}) + (\mathbf{B} \cdot \nabla)\mathbf{v}. \quad (3.7)$$

Where the subscripts "v" and "E" represent the quantity related with velocity and electric field respectively. Then Extended Faraday's law, Eq. (3.6), may be rewritten as,

$$\nabla\times\mathbf{E} = -\mathbf{j_{v-E}} - \frac{\partial \mathbf{B}}{\partial t}. \quad (3.8)$$

The equation of continuity is

$$\nabla \cdot \mathbf{j_{v-E}} = 0. \quad (3.9)$$

The interpretations of terms of the right hand side of Eq. (3.6) are,

1) The term, $\mathbf{B}(\nabla\cdot\mathbf{v})$, describes stretching of the **B** field due to source velocity compressibility;

2) The term, $(\mathbf{B}\cdot\nabla)\mathbf{v}$, describes the stretching or tilting of the **B** field due to the velocity gradients;

For the situation in which, (1) $(\nabla\cdot\mathbf{B})=0$; and (2) the velocity is non-spatially-varying, i.e., $\mathbf{B}(\nabla\cdot\mathbf{v})=(\mathbf{B}\cdot\nabla)\mathbf{v}=0$, Extended Faraday's law, Eq. (3.6), reduces to the Faraday's law,

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}, \tag{3.10}$$

where the **E** field is an axial vector field.

Therefore the extended Faraday's law, Eq. (3.1), is consistent with the Faraday's law. Extended Faraday's law predicts that the spatially-varying velocity, such as e-particles distributing and moving in space, terms $\mathbf{B}(\nabla\cdot\mathbf{v})$ and $(\mathbf{B}\cdot\nabla)\mathbf{v}$ induce axial **E** fields.

3.2. Extended Ampere-Maxwell's Law (1)

The combination of UMFE and Coulomb's law leads us to let $\mathbf{G}=\mathbf{E}.$ Then Eq. (2.18) of UMFE becomes,

$$\oint(\mathbf{v}\times\mathbf{E})\cdot d\mathbf{l}+\frac{d}{dt}\iint\mathbf{E}\cdot ds=\iint\left[\mathbf{v}(\nabla\cdot\mathbf{E})+\frac{\partial\mathbf{E}}{\partial t}-\mathbf{E}(\nabla\cdot\mathbf{v})+(\mathbf{E}\cdot\nabla)\mathbf{v}\right]\cdot ds. \tag{3.11}$$

3.3. Type-2 Duality between Electric and Magnetic Fields

The FAF $(\mathbf{v}\times\mathbf{E})$ and Eq. (3.13) is the type-2 dual of the SAF $(\mathbf{v}\times\mathbf{B})$ and Eq. (3.1). Moreover, Eq. (3.5) is equivalent to Eq. (3.1). Based on the Transfer Rules between dualities of Section 2.2.2, there is a dual of Eq. (3.5), i.e., under transformation,

$$\mathbf{E}\leftrightarrow\mathbf{B}\ \text{ and }\ \mathbf{B}\leftrightarrow-\mathbf{E},$$

we have a type-2 dual of Eq. (3.5), which is,

$$\oint\mathbf{B}\cdot dl=\iint\left[\mathbf{v}(\nabla\cdot\mathbf{E})+\frac{\partial\mathbf{E}}{\partial t}-\mathbf{E}(\nabla\cdot\mathbf{v})+(\mathbf{E}\cdot\nabla)\mathbf{v}\right]\cdot ds, \tag{3.12}$$

which should be equivalent to Eq. (3.11).

With distinguishable feature of "type-1 duality" and "type-2 duality", the duality between induced electric field determined by the Faraday's law and magnetic field determined by Ampere-Maxwell's equation is actually a type-2 duality. UMFE provide the mathematical originations of the type-2 duality between axial electric field and magnetic field.

3.4. Extended Ampere-Maxwell's Law (2)

Eq. (3.12) gives Extended Ampere-Maxwell law,

$$\nabla\times\mathbf{B}=\mathbf{v}(\nabla\cdot\mathbf{E})+\frac{\partial\mathbf{E}}{\partial t}-\mathbf{E}(\nabla\cdot\mathbf{v})+(\mathbf{E}\cdot\nabla)\mathbf{v}. \tag{3.13}$$

The magnetic field **B** is, in the laboratory frame,

$$\mathbf{B}=\mathbf{B}'+\mathbf{v}\times\mathbf{E}. \tag{3.14}$$

Where $\mathbf{B}'$ is the magnetic field at the circuit **dl** in a reference frame in which **dl** is at rest. The **E** is an electric field at the neighborhood of the circuit. The **v** is the velocity of the circuit relative to a laboratory frame.

Note there is no negative sign in front of the $\frac{\partial\mathbf{E}}{\partial t}$, because that the time change of the **E** field through the circuit purely induces a magnetic field $\mathbf{B}'$ that does not accumulate e-particles to against the time change of the **E** field.

3.5. Equation of Continuity

Let's define a "current" generating magnetic field, denote it as

$$\mathbf{j}_{\mathbf{v-B}} = 4\pi\rho_e\mathbf{v} - \mathbf{E}(\nabla\cdot\mathbf{v}) + (\mathbf{E}\cdot\nabla)\mathbf{v}. \quad (3.15)$$

Where the subscripts "v" and "B" represent the quantity related with velocity and magnetic field respectively. Then Eq. (3.13) becomes,

$$\nabla\times\mathbf{B} = \mathbf{j}_{\mathbf{v-B}} + \frac{\partial\mathbf{E}}{\partial\mathbf{t}}. \quad (3.16)$$

The current $\mathbf{j}_{\mathbf{v-B}}$ satisfies the equation of continuity,

$$\nabla\cdot\mathbf{j}_{\mathbf{v-B}} + \frac{\partial\rho_e}{\partial\mathbf{t}} = \mathbf{0}. \quad (3.17)$$

For the situation of the non-spatially-varying velocity, i.e.,

$$\mathbf{E}(\nabla\cdot\mathbf{v}) = (\mathbf{E}\cdot\nabla)\mathbf{v} = 0,$$

we have $\mathbf{j}_{\mathbf{v-B}} = 4\pi\rho_e\mathbf{v}$, Extended Ampere-Maxwell equation, Eq. (3.16), reduces to the Ampere-Maxwell law.

Extended Ampere-Maxwell's law, (1) predicts that the products, $\mathbf{E}(\nabla\cdot\mathbf{v})$ and $(\mathbf{E}\cdot\nabla)\mathbf{v}$, induce respectively axial **E** field; (2) provides a mathematical interpretations why and how e-current and displacement e-current induce inevitably magnetic fields.

3.6. Coulomb Force Extended to Lorentz Force

In a reference frame S' in which a test e-particle q_e is at rest, an electric field is denote as **E**'. The force acting on the test e-particle is the Coulomb force,

$$\mathbf{F} = q_e\mathbf{E}'. \quad (3.18)$$

Transferring to a laboratory frame in which the test e-particle is moving with velocity **v**, Eq. (3.4) gives,

$$\mathbf{E}' = \mathbf{E} + \mathbf{v}\times\mathbf{B}, \quad (3.19)$$

where the electric field **E** is measured in the laboratory frame. The **B** is a magnetic field at the neighborhood of the test e-particle. Substituting Eq. (3.19) into Eq. (3.18), the Coulomb force transfers to the Lorentz force measured in the laboratory frame,

$$\mathbf{F} = q_e\mathbf{E} + q_e\mathbf{v}\times\mathbf{B}. \quad (3.20)$$

3.7. Why Static Electric Field so Different from Magnetic Field

In Introduction, we have mentioned several fundamental differences: "A magnetic field is completely different from a static electric field in the following senses: (1) "e-particle" vs. "e-current"; (2) "$\nabla\cdot\mathbf{E}$" vs. "$\nabla\times\mathbf{B}$"; (3) "vector field **E**" vs. "axial vector field **B**"; (4) "$q_e\mathbf{E}$" vs. "$q_e\mathbf{v}\times\mathbf{B}$".

In this section we have explained those differences:

(1) Comparison of Eq. (3.6) and Eq. (3.13) explains the first, second and third differences as the following: e-particle ρ induces **E** via $\nabla\cdot\mathbf{E} = 4\pi\rho$; it is UMFE combining with the Coulomb law that makes the term, $\mathbf{v}(\nabla\cdot\mathbf{E}) = 4\pi\rho\mathbf{v}$, induces **B** field via $\nabla\times\mathbf{B}$; static e-field **E** is a

vector field, and $\mathbf{B}\sim\mathbf{v}\times\mathbf{E}$ is a first level axial vector field. All of above differences comes from a mathematical origin, Eq. (2.1).

(2) Eq. (3.18) and Eq. (3.20) show how the Coulomb force extends to Lorentz force, when a test e-particle is moving.

4. Classical-Spin-Electromagnetics Derived from UMFE and Coulomb's Law

Let's mathematically establish C-Spin-EM and apply it to study classical phenomena related with spin systematically.

4.1. C-Spin-EM

4.1.1. Definitions of Spin-electric Field and Spin-magnetic Field and Experiment

Let's consider a spinning e-particle characterize by electric charge Q_e and spin $\mathbf{S_c}$. UMFE shows that the spin of an arbitrary source induces **W** and **Z** fields. To applying UMFE to the spin of an e-particle, let $\mathbf{G} = \mathbf{E}$, $\mathbf{M} = \mathbf{B}$, and define

$$\mathbf{B_s} \equiv \mathbf{S_c}\times\mathbf{E}, \tag{4.1}$$

$$\mathbf{E_s} \equiv -\mathbf{S_c}\times\mathbf{B}, \tag{4.2}$$

$$\frac{\mathbf{B_s}}{\mathbf{E_s}} = -\frac{\mathbf{S_c}\times\mathbf{E}}{\mathbf{S_c}\times\mathbf{B}}, \tag{4.3}$$

Naming $\mathbf{E_s}$ as spin-electric field, $\mathbf{B_s}$ as spin-magnetic field. Subscript "s" indicates the quantity related to spin. The electric (magnetic) field can be either an externally applied electric (magnetic) field or a local electric (magnetic) field induced by nearby/lattice e-particles in the material. The $\mathbf{B_s}$ and $\mathbf{E_s}$ are type-2 dual to the **B** and **E** fields, respectively**.**

The $\mathbf{B_s}$ and $\mathbf{E_s}$ are SAF and TAF respectively, and mathematically satisfy,

$$\nabla\cdot\mathbf{B_s} = 0, \tag{4.4}$$

$$\nabla\cdot\mathbf{E_s} = 0. \tag{4.5}$$

We still keep those divergence terms in some of equations of C-Spin-EM, as well the magnetic monopole term $\nabla\cdot\mathbf{B}$, which shows the nature and the breaking mechanism of duality.

If there are "spin-charges" and "Spin-monopole", then we have (Appendix),

$$\nabla\cdot\mathbf{B_s} \neq 0, \tag{4.6}$$

$$\nabla\cdot\mathbf{E_s} \neq 0. \tag{4.7}$$

The definitions, Eq. (4.1) and Eq. (4.2), predict two categories of phenomena.

Firstly, by interacting with an electric field **E** (magnetic field **B),** the spin of e-particles induces an effective spin-magnetic field $\mathbf{B_s}$ (effective spin-electric field $\mathbf{E_s}$).

Secondly, the spin of an e-particle in an electric field **E** (magnetic field **B**) will experience an effective spin-magnetic field $\mathbf{B_s}$ (effective spin-electric field $\mathbf{E_s}$).

Remark: the definitions of $\mathbf{B_s}$ and $\mathbf{E_s}$ are conceptually different from that of Q-Spin-EM.

Testing Experiment 1: An e-particle 1, either without spin or with zero net spin outside a material, which induces an electric field that penetrates into the material (Fig.1). A spinning e-particle 2 inside the material will experience an electric field and an effective spin-magnetic field. Also the e-particle 2 induces an effective spin-magnetic field.

If the orientations of spins of e-particles inside the material are aligned, then the induced effective spin-magnetic field can be detected, which will justify Eq. (4.1) and Eq. (4.2).

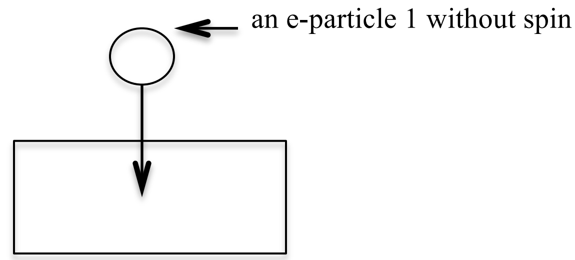

Fig.1

4.1.2. C-Spin-EM Derived from UMFE and Coulomb Law

We can derive C-Spin-EM with three approaches: from Eq. (2.6), from Eq. (2.7), and, based on duality, from Extended EM. The so derived three C-Spin-EMs are mathematically equivalent.

We start with Eq. (2.7). Combination of Eq. (2.7), Eq. (4.1) and Eq. (4.2) gives Ampere-type equations for fields induced by spin of the e-particle respectively,

$$\nabla\times\mathbf{B_s} = \mathbf{S_c}(\nabla\cdot\mathbf{E}) - \boldsymbol{\nabla}(\mathbf{S_c}\cdot\mathbf{E}) - \mathbf{E}[\nabla\cdot\mathbf{S_c}] + 2(\mathbf{E}\cdot\nabla)\mathbf{S_c} + \mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{S_c}) + +\mathbf{S_c}\times(\boldsymbol{\nabla}\times\mathbf{E}), \tag{4.8}$$

$$\nabla\times\mathbf{E_s} = -\mathbf{S_c}(\nabla\cdot\mathbf{B}) + \nabla(\mathbf{S_c}\cdot\mathbf{B}) + \mathbf{B}(\nabla\cdot\mathbf{S_c}) - 2(\mathbf{B}\cdot\nabla)\mathbf{S_c} - \mathbf{B}\times(\nabla\times\mathbf{S_c}) - \mathbf{S_c}\times(\nabla\times\mathbf{B}). \tag{4.9}$$

Substituting Eq. (3.6) and Eq. (3.13) into Eq. (4.8) and Eq. (4.9) respectively, we obtain C-Spin-EM, which includes Ampere-Maxwell-type equation and Faraday-type equations,

$$\nabla\times\mathbf{B_s} = \mathbf{S_c}(\nabla\cdot\mathbf{E}) + \frac{\partial\mathbf{E_s}}{\partial t} - \boldsymbol{\nabla}(\mathbf{S_c}\cdot\mathbf{E}) - \mathbf{E}(\nabla\cdot\mathbf{S_c}) + 2(\mathbf{E}\cdot\nabla)\mathbf{S_c} + \mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{S_c}) + +\mathbf{S_c}\times\{-\mathbf{v}(\nabla\cdot\mathbf{B}) + \mathbf{B}(\nabla\cdot\mathbf{v}) - (\mathbf{B}\cdot\nabla)\mathbf{v}\}, \tag{4.10}$$

$$\nabla\times\mathbf{E_s} = -\mathbf{S_c}(\nabla\cdot\mathbf{B}) - \frac{\partial\mathbf{B_s}}{\partial t} + \nabla(\mathbf{S_c}\cdot\mathbf{B}) + \mathbf{B}(\nabla\cdot\mathbf{S_c}) - 2(\mathbf{B}\cdot\nabla)\mathbf{S_c} - \mathbf{B}\times(\nabla\times\mathbf{S_c}) - -\mathbf{S_c}\times\{\mathbf{v}(\nabla\cdot\mathbf{E}) - \mathbf{E}(\nabla\cdot\mathbf{v}) + (\mathbf{E}\cdot\nabla)\mathbf{v}\}. \tag{4.11}$$

By analogy to "electric current", let's define the "spin-magnetic-current $\mathbf{j_{s-B}}$" and the "spin-electric-current $\mathbf{j_{s-E}}$", which induce the spin-magnetic field and the spin-electric field respectively, as

$$\mathbf{j_{s-B}} \equiv \mathbf{S_c}(\nabla\cdot\mathbf{E}) - \boldsymbol{\nabla}(\mathbf{S_c}\cdot\mathbf{E}) - \mathbf{E}(\nabla\cdot\mathbf{S_c}) + 2(\mathbf{E}\cdot\nabla)\mathbf{S_c} + \mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{S_c}) + +\mathbf{S_c}\times\{-\mathbf{v}(\nabla\cdot\mathbf{B}) + \mathbf{B}(\nabla\cdot\mathbf{v}) - (\mathbf{B}\cdot\nabla)\mathbf{v}\}, \tag{4.12}$$

$$\mathbf{j_{s-E}} \equiv \mathbf{S_c}(\nabla\cdot\mathbf{B}) - \nabla(\mathbf{S_c}\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{S_c}) + 2(\mathbf{B}\cdot\nabla)\mathbf{S_c} + \mathbf{B}\times(\nabla\times\mathbf{S_c}) + +\mathbf{S_c}\times\{\mathbf{v}(\nabla\cdot\mathbf{E}) - \mathbf{E}(\nabla\cdot\mathbf{v}) + (\mathbf{E}\cdot\nabla)\mathbf{v}\}. \tag{4.13}$$

Eq. (4.10) and Eq. (4.11) become respectively,

$$\nabla\times\mathbf{B_s} = \mathbf{j_{s-B}} + \frac{\partial\mathbf{E_s}}{\partial t}, \tag{4.14}$$

$$\nabla\times\mathbf{E_s} = -\mathbf{j_{s-E}} - \frac{\partial\mathbf{B_s}}{\partial t}. \tag{4.15}$$

For the situations, in which spin is non-spatial-varying, i.e.,

$$(\mathbf{E}\cdot\nabla)\mathbf{S_c} = (\boldsymbol{\nabla}\times\mathbf{S_c}) = (\mathbf{B}\cdot\nabla)\mathbf{S_c} = (\nabla\cdot\mathbf{S_c}) = \mathbf{0},$$

Eq. (4.10) to Eq. (4.13) become respectively,

$$\nabla\times\mathbf{B_s} = \mathbf{S_c}(\nabla\cdot\mathbf{E}) + \frac{\partial\mathbf{E_s}}{\partial t} - \boldsymbol{\nabla}(\mathbf{S_c}\cdot\mathbf{E}) + \mathbf{S_c}\times\{-\mathbf{v}(\nabla\cdot\mathbf{B}) + \mathbf{B}(\nabla\cdot\mathbf{v}) - (\mathbf{B}\cdot\nabla)\mathbf{v}\}, \quad (4.16)$$

$$\nabla\times\mathbf{E_s} = -\mathbf{S_c}(\nabla\cdot\mathbf{B}) - \frac{\partial\mathbf{B_s}}{\partial\mathbf{t}} + \nabla(\mathbf{S_c}\cdot\mathbf{B}) - \mathbf{S_c}\times\{\mathbf{v}(\nabla\cdot\mathbf{E}) - \mathbf{E}(\nabla\cdot\mathbf{v}) + (\mathbf{E}\cdot\nabla)\mathbf{v}\}. \quad (4.17)$$

$$\mathbf{j_{s-B}} \equiv \mathbf{S_c}(\nabla\cdot\mathbf{E}) - \boldsymbol{\nabla}(\mathbf{S_c}\cdot\mathbf{E}) + \mathbf{S_c}\times\{-\mathbf{v}(\nabla\cdot\mathbf{B}) + \mathbf{B}(\nabla\cdot\mathbf{v}) - (\mathbf{B}\cdot\nabla)\mathbf{v}\}, \quad (4.18)$$

$$\mathbf{j_{s-E}} \equiv \mathbf{S_c}(\nabla\cdot\mathbf{B}) - \nabla(\mathbf{S_c}\cdot\mathbf{B}) + \mathbf{S_c}\times\{\mathbf{v}(\nabla\cdot\mathbf{E}) - \mathbf{E}(\nabla\cdot\mathbf{v}) + (\mathbf{E}\cdot\nabla)\mathbf{v}\}. \quad (4.19)$$

The $\mathbf{B_s}$ field and $\mathbf{j_{s-B}}$ are type-2 dual of the $\mathbf{E_s}$ field and $\mathbf{j_{s-E}}$ respectively.

For the situations of non-spatial-varying velocity,

$$(\nabla\cdot\mathbf{v}) = (\mathbf{B}\cdot\nabla)\mathbf{v} = (\mathbf{E}\cdot\nabla)\mathbf{v} = \mathbf{0},$$

C-Spin-EM, Eq. (4.16) to Eq. (4.19), are further simplified to,

$$\nabla\times\mathbf{B_s} = 4\pi q_e\mathbf{S_c} + \frac{\partial\mathbf{E_s}}{\partial t} - \boldsymbol{\nabla}(\mathbf{S_c}\cdot\mathbf{E}), \quad (4.20)$$

$$\nabla\times\mathbf{E_s} = -\frac{\partial\mathbf{B_s}}{\partial\mathbf{t}} + \nabla(\mathbf{S_c}\cdot\mathbf{B}) - 4\pi q_e\mathbf{S_c}\times\mathbf{v}. \quad (4.21)$$

$$\mathbf{j_{s-B}} = 4\pi q_e\mathbf{S_c} - \boldsymbol{\nabla}(\mathbf{S_c}\cdot\mathbf{E}), \quad (4.22)$$

$$\mathbf{j_{s-E}} = 4\pi q_e\mathbf{S_c}\times\mathbf{v} - \nabla(\mathbf{S_c}\cdot\mathbf{B}). \quad (4.23)$$

The Coulomb's law, $\nabla\cdot\mathbf{E} = 4\pi q_e$, has been used. Eq. (4.20) and Eq. (4.21) show the following:

(1) The spin, $4\pi q_e\mathbf{S_c}$, and the time change of the $\mathbf{E_s}$ field, $\frac{\partial\mathbf{E_s}}{\partial t}$, induce the $\mathbf{B_s}$ field. Those two terms are the spin-counterparts of e-current and displacement-current respectively.

(2) The gradient of the spin-electric field coupling, $\boldsymbol{\nabla}(\mathbf{S_c}\cdot\mathbf{E})$, induces the $\mathbf{B_s}$ field.

(3) The $\mathbf{E_s}$ field is induced by the time change of the $\mathbf{B_s}$ field, $\frac{\partial\mathbf{B_s}}{\partial t}$, as well $4\pi\frac{q_e}{m_e}\mathbf{S_c}\times\mathbf{p}$.

(4) The gradient of the spin-magnetic field coupling, $\boldsymbol{\nabla}(\mathbf{S_c}\cdot\mathbf{B})$, induces the $\mathbf{E_s}$ field.

Remark: Eq. (4.20) shows that the spin $\mathbf{S_c}$ plays the role of "velocity" generating spin-magnetic field, which is conceptually different from that of Q-Spin-EM.

4.1.3. Equations of Continuity of Spin Currents

The $\mathbf{j_{s-B}}$ and $\mathbf{j_{s-E}}$ should satisfy the equation of continuity respectively. Taking divergence of Eq. (4.14) and Eq. (4.15) respectively, we obtain

$$\nabla\cdot(\nabla\times\mathbf{B_s}) = \nabla\cdot(\mathbf{j_{s-B}}) + \frac{\partial(\nabla\cdot\mathbf{E_s})}{\partial t},$$

$$\nabla\cdot(\nabla\times\mathbf{E_s}) = -\nabla\cdot(\mathbf{j_{s-E}}) - \frac{\partial(\nabla\cdot\mathbf{B_s})}{\partial t}.$$

We face two situations: exist

<u>Firstly</u>. If spin-charge and spin-monopole exist, then $\nabla\cdot\mathbf{E_s} \neq \mathbf{0}$, $\nabla\cdot\mathbf{B_s} \neq \mathbf{0}$. We have familiar form of Equations of Continuity of Spin Currents,

$$\nabla\cdot(\mathbf{j_{s-B}}) + \frac{\partial(\nabla\cdot\mathbf{E_s})}{\partial t} = 0,$$

$$\nabla\cdot(\mathbf{j_{s-E}}) + \frac{\partial(\nabla\cdot\mathbf{B_s})}{\partial t} = 0.$$

<u>Secondly</u>. if we don't have spin-charge and spin-monopoles, equations of continuity are

$$\nabla\cdot(\mathbf{j_{s-B}}) = 0, \quad (4.24)$$

$$\nabla\cdot(\mathbf{j_{s-E}}) = 0, \quad (4.25)$$

To obtain the familiar format of Equations of Continuity, let's take a different approach.

Let's restudy situations, in which, e-particles carry both e-charge and spin, i.e., e-charge and spin are bound together always. Therefore the number density of e-charges is that of spin, namely, the time change and space varying of number density of e-charges are that of spin. Spin current is associated with e-current. The equation of continuity of e-currents, Eq. (3.19), is

$$\nabla \cdot \mathbf{j}_{\mathbf{v-B}} + \frac{\partial \rho_e}{\partial t} = \mathbf{0}, \tag{3.19}$$

$$\mathbf{j}_{\mathbf{v-B}} = 4\pi n q_e \mathbf{v} - \mathbf{E}(\nabla \cdot \mathbf{v}) + (\mathbf{E} \cdot \nabla)\mathbf{v}, \tag{3.17}$$

$$\rho_e = n q_e,$$

where "n" is the number density of e-charge, and thus of spin; q_e is the e-charge of each individual e-particle.

To get the equations of continuity of spin currents, we propose to attach spin to velocity and to convert e-charge density ρ_e to spin density ρ_s. Eq. (3.19) and Eq. (3.17) give

$$\nabla \cdot \mathbf{j}_{s-B} + \frac{\partial \rho_s}{\partial t} = \mathbf{0}, \tag{4.26}$$

where,

$$\mathbf{j}_{\mathbf{s-B}} = 4\pi n q_e \mathbf{S}_c \mathbf{v} - \mathbf{E}[\nabla \cdot (\mathbf{S}_c \mathbf{v})] + (\mathbf{E} \cdot \nabla)(\mathbf{S}_c \mathbf{v}), \tag{4.27}$$

$$\rho_s = \mathbf{S}_c q_e n. \tag{4.28}$$

Note spin $\mathbf{S}_c$ has different orientations, the term, $4\pi n \mathbf{S}_c \mathbf{v}$, need to be expressed as a classical pseudo-tensor spin-magnetic-current,

$$j_{s-B,ij} = 4\pi q_e n v_i S_{cj} - E_i\left[\nabla_k \cdot (\mathbf{S}_c \mathbf{v})_{jk}\right] + (\mathbf{E} \cdot \nabla)(\mathbf{S}_c \mathbf{v})_{ij}, \tag{4.29}$$

$$\rho_{si} = S_{ci} q_e n. \tag{4.30}$$

The generally accepted definition of the spin current pseudo-tensor [3] is,

$$j_{ij} \sim \frac{1}{2}\{S_j v_i + v_i S_j\}. \tag{4.31}$$

Remark: the classical pseudo-tensor spin current represented by Eq. (4.29) is a classical counterpart of the spin current pseudo-tensor represented by Eq. (4.31).

4.1.4. Scalar and Vector Potentials

Based on duality, defining spin-scalar-potential, φ_s, and spin-vector-potential, $\mathbf{A}_s$, as,

$$\mathbf{E}_s \equiv -\nabla \varphi_s - \frac{\partial \mathbf{A}_s}{\partial t}, \tag{4.32}$$

$$\mathbf{B}_s \equiv \nabla \times \mathbf{A}_s. \tag{4.33}$$

Under the gauge transformation,

$$\mathbf{A}_s \to \mathbf{A}_s + \nabla \Lambda_s, \ \ \varphi_s \to \varphi_s - \frac{\partial \Lambda_s}{\partial t}, \tag{4.34}$$

the spin-electric and spin-magnetic fields, $\mathbf{E}_s$ and $\mathbf{B}_s$, are invariant.

C-Spin-EM potentials can be written in terms of EM potentials. Combining Eq. (4.2), Eq. (4.1), Eq. (4.32) and Eq. (4.33), we obtain

$$\nabla \times \mathbf{A}_s = -\mathbf{S}_c \times \nabla \varphi - \mathbf{S}_c \times \frac{\partial \mathbf{A}}{\partial t}, \tag{4.35}$$

$$-\nabla \varphi_s - \frac{\partial \mathbf{A}_s}{\partial t} = -\mathbf{S}_c \times (\nabla \times \mathbf{A}). \tag{4.36}$$

4.1.5. Spin Wave

C-Spin-EM predicts classical spin waves described by,

$$\frac{\partial^2 \mathbf{B_s}}{\partial t^2} - \nabla^2 \mathbf{B_s} = \nabla\times\mathbf{j_{s-B}} - \frac{\partial}{\partial t}\mathbf{j_{s-E}}, \tag{4.37}$$

$$\frac{\partial^2 \mathbf{E_s}}{\partial t^2} - \nabla^2 \mathbf{E_s} = -\nabla\times\mathbf{j_{s-E}} - \frac{\partial \mathbf{j_{s-B}}}{\partial t}. \tag{4.38}$$

Remark: (1) By duality between EM and C-Spin-EM, spin waves can be quantized and the quanta are spin-one Bosons. (2) The propagation speed of spin wave is to be determined.

4.2. C-Spin-EM vs. Q-Spin-EM

Let's study the similarity and difference between C-Spin-EM and Q-Spin-EM. To convert to quantum theory, we need to introduce the concept of phase. Eq. (4.32) and Eq. (4.33) gives,

$$\oint \mathbf{E}_s \cdot \mathrm{dl} = -\frac{\partial}{\partial t}\oint \mathbf{A}_s \cdot \mathrm{d}\mathbf{l}, \tag{4.39}$$

$$\iint \mathbf{B_s} \cdot \mathbf{ds} = \oint \mathbf{A}_s \cdot \mathrm{d}\mathbf{l}. \tag{4.40}$$

We define,

$$\oint \mathbf{A}_s \cdot \mathrm{d}\mathbf{l} \equiv \Phi_s, \tag{4.41}$$

then have

$$\Phi_s = \iint \mathbf{B_s} \cdot \mathbf{ds}, \tag{4.42}$$

$$\dot{\Phi}_s = -\oint \mathbf{E}_s \cdot \mathrm{d}\mathbf{l}. \tag{4.43}$$

Defining Φ_s as the phase, $e^{i\Phi_s}$, spin-electric field $\mathbf{E}_s$ and spin-magnetic field $\mathbf{B}_s$ of C-Sin-EM have the same form as that of Spin motive force $\mathbf{E}_{Qs}$ and Berry curvature $\mathbf{B}_{Qs}$ of Q-Spin-EM.

However, the fundamental differences between C-Spin-EM and Q-Spin-EM are the definitions of spin-electric field and spin-magnetic field, as well the field equations.

There are analogies between quantities of EM and quantum anholonomy. Since the type-2 dualities between EM and C-Spin-EM, we propose that there are analogies between quantities of C-Spin-EM and quantum anholonomy, Table 1.

Table 1: Analogies

Quantum Anholonomy	EM	C-Spin-EM
Berry connection	$\mathbf{A}$	$\mathbf{A_s}$
Berry curvature	$\mathbf{B}$	$\mathbf{B}_s$
Berry phase	Magnetic flux	Spin-magnetic flux Φ_s

4.3. Lagrangian and Hamiltonian

For a non-relativistic non-spinning e-particle Q_e in EM field, the regular Lagrangian and Hamiltonian are respectively,

$$\mathcal{L}_{reg} = \frac{1}{2}mv^2 + Q_e\mathbf{A}\cdot\mathbf{v} - Q_e\varphi,$$

$$H_{reg} = \frac{1}{2m}(p - Q_e\mathbf{A})^2 + Q_e\varphi. \tag{4.44}$$

For a spinning e-particle in C-Spin-EM fields, the Lagrangian should contain its rotation energy, $KE_{spin} = \frac{1}{2}I\omega^2$. Defining $a_{\mathcal{L}} \equiv \frac{I\omega^2}{(\mathbf{S}_c)^2}$, we have

$$KE_{spin} \equiv \frac{1}{2}a_{\mathcal{L}}(\mathbf{S}_c)^2, \tag{4.45}$$

By the duality between Extended EM and C-Spin-EM, let's introduce Lagrangian,

$$\mathcal{L}_{spin} = \frac{1}{2}a_{\mathcal{L}}(\mathbf{S_c})^2 + Q_e\mathbf{A_s}\cdot\mathbf{S_c} - Q_e\varphi_s. \tag{4.46}$$

Taking into account the interaction between the velocity and spin-vector-potential, and between the spin and vector potential, we obtain

$$\mathcal{L}_{inter} = Q_e\mathbf{A_s}\cdot\mathbf{v} + Q_e\mathbf{A}\cdot\mathbf{S_c}. \tag{4.47}$$

The total Lagrangian of a spinning e-particle in EM and C-Spin-EM fields is

$$\mathcal{L}_{total} = \frac{1}{2}mv^2 + Q_e\mathbf{A}\cdot\mathbf{v} - Q_e\varphi + \frac{1}{2}a_{\mathcal{L}}(\mathbf{S_c})^2 + Q_e\mathbf{A_s}\cdot\mathbf{S_c} - Q_e\varphi_s + \\ +Q_e\mathbf{A_s}\cdot\mathbf{v} + Q_e\mathbf{A}\cdot\mathbf{S_c}. \tag{4.48}$$

In the derivation of C-Spin-EM, we have replace velocity **v** by spin $\mathbf{S_c}$ in UMFE. Now we use spin as a "generalized velocity", substituting it into Hamiltonian,

$$H = \sum \dot{q}^i \frac{\partial\mathcal{L}_{total}}{\partial\dot{q}^i} - \mathcal{L}_{total} = v\frac{\partial\mathcal{L}_{total}}{\partial v} + \mathbf{S_c}\frac{\partial\mathcal{L}_{total}}{\partial\mathbf{S_c}} - \mathcal{L}_{total}, \tag{4.49}$$

we obtain the Hamiltonian for spinning e-particles,

$$H = \frac{(\mathbf{p} - Q_e\mathbf{A} - Q_e\mathbf{A_s})^2}{2m} + \frac{(\mathbf{p}_s - Q_e\mathbf{A}_s - Q_e\mathbf{A})^2}{2a_{\mathcal{L}}} + Q_e\varphi + Q_e\varphi_s, \tag{4.50}$$

which describes dynamics of spinning e-particles in both Extended EM and C-Spin-EM fields. Where the $\mathbf{p}_s$ is a conjugate momentum corresponding to the "generalized velocity" $\mathbf{S_c}$,

$$\mathbf{p}_s = \frac{\partial\mathcal{L}_{total}}{\partial\mathbf{S_c}}. \tag{4.51}$$

Next we will study the effects of the following terms of Eq. (4.50),

$$\frac{(\mathbf{p} - Q_e\mathbf{A} - Q_e\mathbf{A_s})^2}{2m} \approx \frac{(\mathbf{p})^2}{2m} - \frac{Q_e\mathbf{p}\cdot\mathbf{A}}{m} - \frac{Q_e\mathbf{p}\cdot\mathbf{A_s}}{m}, \tag{4.52}$$

$$\frac{(\mathbf{p}_s - Q_e\mathbf{A}_s - Q_e\mathbf{A})^2}{2a_{\mathcal{L}}} \approx \frac{(\mathbf{p}_s)^2}{2a_{\mathcal{L}}} - \frac{Q_e\mathbf{p}_s\cdot\mathbf{A}_s}{a_{\mathcal{L}}} - \frac{Q_e\mathbf{p}_s\cdot\mathbf{A}}{a_{\mathcal{L}}}. \tag{4.53}$$

where non-linear terms have been ignored.

In the following applications, both uniform magnetic field **B** and uniform spin-magnetic field $\mathbf{B}_s$ are in z-direction, vector potential **A** and spin-vector-potential $\mathbf{A}_s$ have similar form,

$$\mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{r}, \tag{4.54}$$

$$\mathbf{A}_s = -\mathbf{B}_s\times\mathbf{r} = -(\mathbf{S_c}\times\mathbf{E})\times\mathbf{r}. \tag{4.55}$$

Eq. (4.55) shows the relation between spin-vector potential and spin-magnetic field induced by spin, which has the same form as that induced by magnetic momentum [4].

Remark: With Hamiltonian of Eq. (4.50), C-spin-EM can be converted to its quantum version. The Hamiltonian not only provides classical counterparts/origins of several quantum phenomena, but also predicts several classical effects that may be converted to quantum effects.

4.4. Effects of Hamiltonian

4.4.1. Extended-Rashba-SOC-1, Spin-Zeeman Effect and Experiment

Rashba SOC is a fundamental effect. Let's extend Rashba SOC. Substituting Eq. (4.55) into the third term of Eq. (4.52), we obtain,

$$-\frac{Q_e \mathbf{p}\cdot\mathbf{A_s}}{m} = \frac{Q_e}{m}\mathbf{p}\cdot(\mathbf{B}_s\times\mathbf{r}) = \frac{Q_e}{m}\mathbf{B}_s\cdot\mathbf{L}. \quad (4.56)$$

With the definitions of spin-magnetic field, $\mathbf{B}_s = \mathbf{S_c}\times\mathbf{E}$, let's re-write Eq. (4.56), denote as H_{SOC-1},

$$H_{SOC-1} = \frac{Q_e}{m}\mathbf{B}_s\cdot\mathbf{L} = \frac{Q_e}{m}\mathbf{E}\cdot(\mathbf{L}\times\mathbf{S_c}). \quad (4.57)$$

We refer Eq. (4.57) as Extended-Rashba-SOC-1. Comparing with $H_{Rashba} = \alpha_R\mathbf{E}\cdot(\boldsymbol{\sigma}\times\mathbf{p})$.

Remark: (1) The **p** represents a linear motion; $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ represents an orbiting motion; thus the term, $\mathbf{L}\times\mathbf{S}_c$, represents indeed a Spin-Orbit-coupling. Actually, when Rashba SOC is applied to several situations, the momentum **p** is replaced by angular momentum **L** [5]. (2)

We will show that, in Eq. (4.115), the spin-magnetic force causes H_{SOC}.

Zeeman Effect,

$H_{Zeeman} = -\gamma\mathbf{L}\cdot\mathbf{B}$

has important applications. The second term of Hamiltonian, Eq. (4.52), causes the regular Zeeman effect. The third term causes an additional shift, denoted as spin-Zeeman effect,

$$H_{spin-Z} = -\frac{Q_e \mathbf{p}\cdot\mathbf{A_s}}{m} = \frac{Q_e}{m}\mathbf{B}_s\cdot\mathbf{L}, \quad (4.58)$$

which represents the interaction between a $\mathbf{B}_s$ field and orbiting motion.

Testing Experiment 2: The spin-Zeeman effect is identical to Extended-Rashba-SOC-1, which provides a test that by measuring the spin-Zeeman shift one can test Extended-Rashba-SOC-1.

4.4.2. Extended-Rashba-SOC-2

Substituting Eq. (4.55) into the second term of Eq. (4.53), we obtain Extended-Rashba-SOC-2, denote as H_{SOC-2},

$$H_{SOC-2} = -\frac{Q_e \mathbf{p}_s\cdot\mathbf{A}_s}{a_{\mathcal{L}}} = \frac{Q_e}{a_{\mathcal{L}}}\mathbf{p}_s\cdot(\mathbf{B}_s\times\mathbf{r}) = \frac{Q_e}{a_{\mathcal{L}}}\mathbf{E}\cdot(\mathbf{L}_s\times\mathbf{S_c}), \quad (4.59)$$

where $\mathbf{L}_s$ is defined as

$$\mathbf{L}_s \equiv \mathbf{r}\times\mathbf{p}_s, \quad (4.60)$$

called "conjugate angular momentum" corresponding to conjugate momentum $\mathbf{p}_s$.

4.4.3. Extended-Rashba-SOC-3

An orbiting spinning particle has angular momentum **L** and conjugate angular momentum $\mathbf{L}_s$ that contains spin $\mathbf{S}_c$. To derive a total angular momentum in C-Spin-EM, combining Eq. (4.58) and Eq. (4.59), we define a total angular momentum **J** and Hamiltonian for Extended-Rashba-SOC-3,

$$\mathbf{J} \equiv \frac{Q_e}{m}\mathbf{L} + \frac{Q_e}{a_{\mathcal{L}}}\mathbf{L}_s. \quad (4.61)$$

$$H_{SOC-3} \equiv -\mathbf{B}_s\cdot\mathbf{J} = -(\mathbf{S_c}\times\mathbf{E})\cdot\mathbf{J} = \mathbf{E}\cdot(\mathbf{S_c}\times\mathbf{J}). \quad (4.62)$$

We refer it as Extended-Rashba-SOC-3.

4.4.4. Conjugate Angular Momentum-Magnetic Field Coupling

Combining Eq. (4.54) and the term, $\frac{Q_e \mathbf{p_s} \cdot A}{a_{\mathcal{L}}}$, of Eq. (4.53), we obtain Hamiltonian for Conjugate Angular Momentum-Magnetic Field Coupling, denoted as $H_{\mathbf{L_s}-\mathbf{B}}$,

$$H_{\mathbf{L_s}-\mathbf{B}} = \frac{Q_e \mathbf{p_s} \cdot \mathbf{A}}{\mathbf{a}_{\mathcal{L}}} = \frac{Q_e \mathbf{p_s} \cdot (\mathbf{B} \times \mathbf{r})}{2a_{\mathcal{L}}} = \frac{Q_e \mathbf{B} \cdot (\mathbf{r} \times \mathbf{p_s})}{2a_{\mathcal{L}}} = \frac{Q_e}{2a_{\mathcal{L}}} \mathbf{B} \cdot \mathbf{L}_s. \quad (4.63)$$

4.4.5. Total angular Momentum-Magnetic Field Coupling

Combining $H_{\mathbf{L_s}-\mathbf{B}}$ with the second term of Eq. (4.52), $\frac{Q_e \mathbf{p} \cdot \mathbf{A}}{m}$, and Eq. (4.61), we obtain the Total angular Momentum-Magnetic Field Coupling,

$$H_{B-\text{total}} = \frac{1}{2} \mathbf{B} \cdot \mathbf{J}. \quad (4.64)$$

4.4.6. Spin-Aharonov–Bohm Effect and Experiment

The regular Hamiltonian, Eq. (4.44), causes the phase shift of Aharonov–Bohm effect. Eq. (4.50) predicts an effect that a spin-vector-potential $\mathbf{A}_s$ induces a phase shift, $\Delta\varphi_{\text{spin}}$,

$$\Delta\varphi_{\text{spin}} \sim \frac{Q_e}{\hbar} \oint A_s \cdot d\mathbf{r}, \quad (4.65)$$

$$\oint A_s \cdot d\mathbf{r} = \iint \mathbf{B_s} \cdot d\mathbf{s} = \iint (\mathbf{S_c} \times \mathbf{E}) \cdot d\mathbf{s} = \iint \left\{ \mathbf{S_c} \times \left(-\nabla\varphi_{\mathbf{e}} - \frac{\partial \mathbf{A}}{\partial t} \right) \right\} \cdot d\mathbf{s}. \quad (4.66)$$

which we denote as the Spin-Aharonov–Bohm effect, which is caused by the interaction between e-particles' spin, gradient of electric scalar potential, $\nabla\varphi_{\mathbf{e}}$, and time changing of magnetic vector potential, $\frac{\partial \mathbf{A}}{\partial t}$.

When a spinning e-particle travelling along the same path P in a region with non-zero **A**, $\nabla\varphi_{\mathbf{e}}$ and $\frac{\partial \mathbf{A}}{\partial t}$, acquires a total phase shift, $\Delta\varphi_{\text{total}}$, which extends Aharonov–Bohm effect,

$$\Delta\varphi_{\text{total}} = \Delta\varphi_{AB} + \Delta\varphi_{\text{spin}} = \frac{Q_e}{\hbar} \oint \mathbf{A} \cdot d\mathbf{r} + \frac{Q_e}{\hbar} \oint \mathbf{A}_s \cdot d\mathbf{r}. \quad (4.67)$$

Testing Experiment 3: In the Aharonov–Bohm double-slit experiment, change the vector potential **A** with time.

Remark: Eq. (4.66) shows that the magnitude of a Spin-Aharonov–Bohm effect depends on the relative directions between $\mathbf{S_c}$, $\nabla\varphi_{\mathbf{e}}$, and $\frac{\partial \mathbf{A}}{\partial t}$.

4.5. Spin-Lorentz-type Force

Based on the type-2 duality between Extended EM and C-Spin-EM, we postulate that, beside the dualities between fields and field equations, there is a type-2 duality between Lorentz force and a force, denoted as spin-Lorentz-type force, i.e., under the transformation,

$$\mathbf{E} \leftrightarrow \mathbf{E_s}, \ \mathbf{B} \leftrightarrow \mathbf{B_s}$$

Lorentz force, $\mathbf{F}_{\text{Lorentz}}$ (abbreviate $\mathbf{F_L}$), converts to Spin-Lorentz-type force, $\mathbf{F}_{\text{spin-Lorentz}}$ (abbreviate $\mathbf{F}_{SL}$), and vice versa,

$$\mathbf{F}_{SL} = m\frac{dv}{dt} = Q_e \mathbf{E_s} + Q_e \mathbf{v} \times \mathbf{B_s}. \quad (4.68)$$

The "**v**" is the velocity of a test e-particle. We refer "$Q_e \mathbf{E_s}$" as spin-electric force, "$Q_e \mathbf{v} \times \mathbf{B_s}$" as spin-magnetic force.

A moving spinning e-particle Q_e experiences both Lorentz force and Spin-Lorentz-type forces, denoted as Total-Lorentz-type force, $\mathbf{F}_{\text{Total-Lorentz}}$ (abbreviated $\mathbf{F}_{TL}$),

$$\mathbf{F}_{TL} = \mathbf{F_L} + \mathbf{F_{sL}} = Q_e \mathbf{E} + Q_e \mathbf{v} \times \mathbf{B} + Q_e \mathbf{E_s} + Q_e \mathbf{v} \times \mathbf{B_s} \quad (4.69)$$

Using definitions of $\mathbf{E_s}$ and $\mathbf{B_s}$ in terms of electric **E** and magnetic **B** fields, we obtain,

$$\mathbf{F}_{\mathrm{TL}} = \mathrm{Q_e}\mathbf{E} + \mathrm{Q_e}\mathbf{v}\times\mathbf{B} - \mathrm{Q_e}[\mathrm{a}\mathbf{S_c}\times\mathbf{B}] + \mathrm{Q_e}\mathbf{v}\times[\mathrm{a}\mathbf{S_c}\times\mathbf{E}]. \quad (4.70)$$

The "a" is a coefficient, such that $\mathrm{Q_e}[(\mathrm{a}\mathbf{S_c})\times\mathbf{B}]$ and $\mathrm{Q_e}\mathbf{v}\times[(\mathrm{a}\mathbf{S_c})\times\mathbf{E}]$ have the unit of force. Here after, absorbing "a" into $\mathbf{S_c}$.

4.6. Extended Landau–Lifshitz and Landau–Lifshitz-Gilbert Equations

Moreover, base on the type-2 duality, under the transformation,

$$\mathbf{v} \leftrightarrow \mathbf{S_c},\ \ \mathbf{E} \leftrightarrow \mathbf{E_s},\ \ \mathbf{B} \leftrightarrow \mathbf{B_s},$$

Lorentz force equation converts to a Landau–Lifshitz-type equation,

$$\mathrm{m}\frac{\mathrm{d}\mathbf{S_c}}{\mathrm{dt}} = \mathrm{Q_e}\mathbf{E_s} + \mathrm{Q_e}\mathbf{S_c}\times\mathbf{B_s} = -\mathrm{Q_e}\mathbf{S_c}\times\mathbf{B} + \mathrm{Q_e}\mathbf{S_c}\times(\mathbf{S_c}\times\mathbf{E}). \quad (4.71)$$

Which predicts that not only magnetic field but also electric field induces spin precession, which is a counterpart of gyroscope precession in gravitational field. Combining Eq. (4.71) with LL and LLG equations respectively, we obtain Extended LL and Extended LLG equations,

$$\frac{\mathrm{d}\mathbf{M}}{\mathrm{dt}} = -\gamma\mathbf{M}\times\mathbf{H}_{\mathrm{eff}} - \lambda\mathbf{M}\times(\mathbf{M}\times\mathbf{H}_{\mathrm{eff}}) + \alpha\mathbf{M}\times(\mathbf{M}\times\mathbf{D}_{\mathrm{eff}}), \quad (4.72)$$

$$\frac{\mathrm{d}\mathbf{M}}{\mathrm{dt}} = -\gamma\mathbf{M}\times\mathbf{H}_{\mathrm{eff}} + \beta\mathbf{M}\times\frac{\mathrm{d}\mathbf{M}}{\mathrm{dt}} + \alpha\mathbf{M}\times(\mathbf{M}\times\mathbf{D}_{\mathrm{eff}}). \quad (4.73)$$

Where the spin has been replaced by the magnetization **M**, and $\mathbf{B} \rightarrow \mathbf{H}_{\mathrm{eff}}$, $\mathbf{E} \rightarrow \mathbf{D}_{\mathrm{eff}}$; the "α" is coefficient.

4.7. Effects of Spin-Lorentz-type Force

4.7.1. Dual-Hall Effect/Topological Insulator and Experiment

The term, "$\mathrm{Q_e}\mathbf{S_c}\times\mathbf{B}$", of Eq. (4.70) causes a new effect that a magnetic field **B** in z-direction acting on the spin of e-particles, even the centers of the spinning e-particles are originally at rest, drives e-particles to move, which causes e-particles accumulation at the opposite surrounding edges, which in turn causes transverse electric fields, $\mathrm{E_x}$ and $\mathrm{E_y}$. Assuming the magnetic field **B** is in z-direction, at equilibrium, $\mathrm{v_x} = \mathrm{v_y} = 0$, we have,

$$\mathrm{E_x} = \mathrm{S_{cy\pm}}\mathrm{B_z}, \quad (4.74)$$

$$\mathrm{E_y} = -\mathrm{S_{cx\pm}}\mathrm{B_z}. \quad (4.75)$$

Comparing with Hall transverse electric field, $\mathrm{E_y} = \mathrm{v_x}\mathrm{B_z}$, we refer this effect as the Dual-Hall Effect, i.e., the regular transverse Hall electric field converts to the transverse Dual-Hall electric field, and vice versa. To detect the Dual-Hall Effect is to test the spin-Lorentz-type force.

Testing Experiment 4: Place a sheet of material in a magnetic field in z-direction, measuring transverse electric fields without applying an external electric field.

Remark: The fundamental differences are: (1) In Hall effect, the motion of e-particles is required, while not required for Dual-Hall effect; (2) In Hall effect, the transverse electric field points to one direction, say either $+\mathrm{y}$ or $-\mathrm{y}$, while in Dual-Hall effect, transverse electric fields are in two directions, $\pm\mathrm{x}$ and $\pm\mathrm{y}$, depend on the orientations of spin, which causes topological insulator.

4.7.2. Extended-Hall Effect/Topological Insulator

Eq. (4.70) shows that the spin-magnetic force, $Q_e\{\mathbf{v}\times(\mathbf{S_c}\times\mathbf{E})\}$, deflects the trajectory of moving spinning e-particles, which causes the buildup of e-particles on opposite surrounding edge-surfaces. The buildup induces transverse electric fields that balance the spin-magnetic forces, $Q_e\mathbf{E} = -Q_e\{\mathbf{v}\times(\mathbf{S_c}\times\mathbf{E})\}$.

For classical spin there is no restriction on orientation of spin. Eq. (4.70) indicates that the electric field has not effect on spins that are in the same direction; thus, one can say that there is no spin in the applied electric field/current direction, or one only needs to consider the spins with orientations perpendicular to the direction of applied electric field. However, in our case, there is longitudinal electric field in x-direction, and transverse electric fields in both y- and z-directions, thus we need to consider spins in all x-, y-, and z-directions. We still use the term "spin" to represent the intrinsic angular momentum including those in the same direction of movement of e-particles. When convert to quantum, the concepts of Chirality and Helicity appear.

The spin-Lorentz-type force is the spin's orientation-dependent. Let's consider random equal distribution of spin. For simplicity, denote spins with orientations along positive/negative x-axis, y-axis, and z-axis, as, respectively, $S_{cx\pm}$, $S_{cy\pm}$ and $S_{cz\pm}$. The positive/negative sings "$\pm$" refer to spin-up/spin-down, S_{cx+}/S_{cx-}, S_{cy+}/S_{cy-}, S_{cz+}/S_{cz-}, in that direction, respectively.

In regular Hall experiment, both an external magnetic field **B** and an external electric field **E** are applied simultaneously. Now we study a 3D material placed in both a **B** field (z-axis) and an **E** field (x-axis) that drives a longitudinal current density j_x flowing along x-axis (Fig. 2). Spin $\mathbf{S_c}$ can be either in $\pm x$ direction, or $\pm y$ direction, or $\pm z$ direction, denoted as 3-current model.

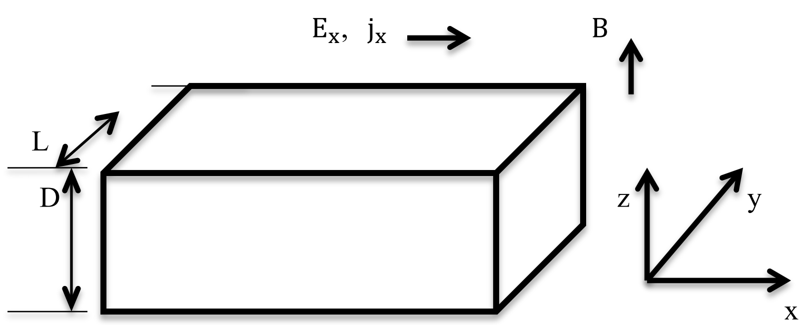

Fig. 2

Starting with the Total-Lorentz-type force, $\mathbf{F}_{TL}$. An e-current in one direction contains e-particles with spins in different orientations. We obtain equations of motion, respectively,

$$\frac{mv_x}{Q_e\tau} = E_x - S_{cy\pm}B_z,$$

$$\frac{mv_y}{Q_e\tau} = E_y - v_xB_z + S_{cx\pm}B_z - v_xS_{cx\pm}E_y + v_xS_{cy\pm}E_x,$$

$$\frac{mv_z}{Q_e\tau} = E_z - v_xS_{cx\pm}E_z + v_xS_{cz\pm}E_x.$$

And obtain, in equilibrium, $v_y = v_z = 0$,

$$v_x = \frac{\tau Q_e}{m}\{E_x - S_{cy\pm}B_z\}, \tag{4.76}$$

$$E_x = \frac{mv_x}{Q_e\tau} + S_{cy\pm}B_z, \tag{4.77}$$

$$E_y = \frac{v_xB_z - S_{cx\pm}B_z - v_xS_{cy\pm}E_x}{(1 - v_xS_{cx\pm})}, \tag{4.78}$$

$$E_z = -\frac{v_x(S_{cz\pm}E_x)}{(1-v_xS_{cx\pm})}. \tag{4.79}$$

The induced transverse electric fields depend on orientations of spins. The spinning e-particles are driven to surrounding edge/surfaces at $\pm$y and $\pm$z symmetrically. The accumulations make edge/surfaces having better conductivity than bulk.

We use the same definitions of Hall coefficients R_{c-ij} and resistivity ρ_{c-ij}, and obtain

$$R_{ext-xx} \equiv \frac{E_x}{j_xB_z} = \frac{m}{n\tau Q_e^2B_z} + \frac{mS_{cy\pm}}{n\tau Q_e^2(E_x - S_{cy\pm}B_z)}, \tag{4.80}$$

$$\rho_{ext-xx} \equiv R_{ext-xx}B_z = \frac{m}{n\tau Q_e^2} + \frac{mS_{cy\pm}B_z}{n\tau Q_e^2(E_x - S_{cy\pm}B_z)}, \tag{4.81}$$

$$R_{ext-yx} \equiv \frac{E_y}{j_xB_z} = \frac{1}{nQ_e(1-v_xS_{cx\pm})}\left\{1 - \frac{S_{cx\pm}}{v_x} - S_{cy\pm}\left(\frac{E_x}{B_z}\right)\right\}, \tag{4.82}$$

$$\rho_{ext-xy} \equiv R_{ext-yx}B_z = \frac{B_z}{nQ_e(1-v_xS_{cx\pm})}\left\{1 - \frac{S_{cx\pm}}{v_x} - S_{cy\pm}\left(\frac{E_x}{B_z}\right)\right\}, \tag{4.83}$$

$$R_{ext-zx} \equiv \frac{E_z}{j_xB_z} = -\frac{(S_{cz\pm})}{nQ_e(1-v_xS_{cx\pm})}\left(\frac{E_x}{B_z}\right), \tag{4.84}$$

$$\rho_{ext-xz} \equiv R_{ext-zx}B_z = -\frac{(S_{cz\pm})E_x}{nQ_e(1-v_xS_{cx\pm})}. \tag{4.85}$$

We refer the effect described by Eq. (4.76) to Eq. (4.85) as *Extended-Hall effect/Topological insulator,* which is caused by Total-Lorentz-type force.

Remark: The term, "$\mathbf{v}\times(\mathbf{S}_c\times\mathbf{E})$", is the classical origin of that no magnetic field required in Spin-Hall effect of quantum**,** but an electric field, either an external or a local, is required.

4.7.3. Extended-Hall effect having Zero Longitudinal Hall Coefficient/Resistivity

For strong magnetic field, $E_x \ll S_{cy\pm}B_z$, Eq. (4.80) to Eq. (4.83) reduce to,

$$R_{ext-xx} = \frac{m}{n\tau Q_e^2B_z} - \frac{m}{nQ_e^2\tau B_z\left\{1 - \frac{E_x}{S_{cy\pm}B_z}\right\}} \approx 0, \tag{4.86}$$

$$\rho_{ext-xx} = \frac{m}{n\tau Q_e^2} - \frac{m}{nQ_e^2\tau\left\{1 - \frac{E_x}{S_{cy\pm}B_z}\right\}} \approx 0, \tag{4.87}$$

$$R_{ext-yx} \approx \frac{1}{nQ_e(1-v_xS_{cx\pm})}\left\{1 + \frac{m}{\tau Q_eB_z}\left(\frac{S_{cx\pm}}{S_{cy\pm}}\right) - S_{cy\pm}\left(\frac{E_x}{B_z}\right)\right\}, \tag{4.88}$$

$$\rho_{ext-xy} \approx \frac{B_z}{nQ_e(1-v_xS_{cx\pm})} + \frac{m}{n\tau Q_e^2(1-v_xS_{cx\pm})}\left(\frac{S_{cx\pm}}{S_{cy\pm}}\right) - \frac{E_xS_{cy\pm}}{nQ_e(1-v_xS_{cx\pm})}. \tag{4.89}$$

For $1 \gg v_xS_{cx\pm}$, we obtain

$$R_{ext-yx} \approx \frac{1}{nQ_e} + \frac{m}{n\tau Q_e^2B_z}\left(\frac{S_{cx\pm}}{S_{cy\pm}}\right) - \frac{S_{cy\pm}}{nQ_e}\left(\frac{E_x}{B_z}\right), \tag{4.90}$$

$$\rho_{ext-xy} \approx \frac{B_z}{nQ_e} + \frac{m}{n\tau Q_e^2}\left(\frac{S_{cx\pm}}{S_{cy\pm}}\right) - \frac{E_xS_{cy\pm}}{nQ_e}. \tag{4.91}$$

$$R_{ext-zx} \approx -\frac{(S_{cz\pm})}{nQ_e}\left(\frac{E_x}{B_z}\right), \tag{4.92}$$

$$\rho_{ext-xz} \approx -\frac{E_x S_{cz\pm}}{nQ_e}. \quad (4.93)$$

Remarks: (1) Eq. (4.76), Eq. (4.86) and Eq. (4.87) show that when spin-electric field, $\mathbf{E}_s = \mathbf{S}_c \times \mathbf{B}$, i.e., strong magnetic field, dominants the longitudinal current, there are zero longitudinal, $R_{ext-xx} = 0$ and $\rho_{ext-xx} = 0$, which is a classical counterpart of Hall conductance quantization in edge state, $R_{H-xx} = 0$ and $\rho_{H-xx} = 0$. (2) For strong longitudinal electric field, we have $\rho_{ext-xx} \neq 0$.

4.7.4. Extended-Hall effect Contributing to GMR/TMR Effect and Experiment

Starting with Extended Hall Resistivity for the 3D model in Fig. 2,

$$\rho_{ext-xx} = \frac{m}{n\tau Q_e^2} + \frac{m S_{cy\pm} B_z}{n\tau Q_e^2 (E_x - S_{cy\pm} B_z)}. \quad (4.81)$$

The relative resistance change is calculated as,

$$\delta\rho_{ext-xx} = \frac{\rho_{ext-xx}(\mathbf{B}) - \rho_{ext-xx}(0)}{\rho_{ext-xx}(0)} = \frac{\frac{m}{n\tau Q_e^2} + \frac{m S_{cy\pm} B_z}{n\tau Q_e^2 (E_x - S_{cy\pm} B_z)} - \frac{m}{n\tau Q_e^2}}{\frac{m}{n\tau Q_e^2}} = \frac{S_{cy\pm} B_z}{E_x - S_{cy\pm} B_z}. \quad (4.94)$$

Starting from zero magnetic field and increases it, $\delta\rho_{ext-xx}$ increases. There is a turning point,

$$E_x = S_{cy\pm} B_z. \quad (4.95)$$

After the turning point, the **B** field continuously increases, we have $E_x < S_{cy\pm} B_z$, thus

$$\rho_{ext-xx}(\mathbf{B}) < \rho_{ext-xx}(0), \quad (4.96)$$

which implies that an external magnetic field **B**, starting at certain strength, decreases the magnetoresistance.

When a situation

$$E_x \ll |S_{cy\pm} B_z|, \quad (4.97)$$

is reached, Eq. (4.94) gives

$$\delta\rho_{ext-xx} \approx -1, \quad (4.98)$$

which implies a Giant/TMR magnetoresistance,

$$\rho_{ext-xx}(\mathbf{B}) \approx 0, \quad (4.99)$$

which agrees with zero longitudinal Hall resistance of Eq. (4.87) and of the quantum Hall effect.

Note, there is always resistance from the insulator layer in GMR/TMR, the net magnetoresistance is equal to that of insulator and, thus is a non-zero constant.

Testing Experiment 5: Reducing insulator layer's resistance to test whether $\rho_{ext-xx}(\mathbf{B}) \rightarrow 0$.

Remark: This derivation of GMR/TMR contributes a mechanism in addition to spin scattering.

4.7.5. Extended-Hall Effect Contributing to Spin Hall Effect

With absence of a magnetic field, there are still transverse dual electric fields, E_y and E_z,

$$E_y = -\frac{v_x S_{cy\pm} E_x}{(1 - v_x S_{cx\pm})} = -\frac{\tau Q_e S_{cy\pm}}{m(1 - v_x S_{cx\pm})} E_x^2, \quad (4.100)$$

$$E_z = -\frac{v_x S_{cz\pm} E_x}{(1 - v_x S_{cx\pm})} = -\frac{\tau Q_e S_{cz\pm}}{m(1 - v_x S_{cx\pm})} E_x^2. \quad (4.101)$$

Remark: the spin-magnetic force, $Q_e\{\mathbf{v}\times(\mathbf{S_c}\times\mathbf{E})\}$**,** contributes to spin Hall effect. In the absence of magnetic field: (1) transverse E_y and E_z are dual/symmetry; an external magnetic field **B** breaks the duality/symmetry; (2) transverse E_y and E_z fields are proportional to the square of longitudinal electric field, which agrees with experiments.

4.7.6. Extended-Hall effect Contributing to Anomalous-Hall Effect

Let's compare with the empirical equation of Anomalous Hall effect,

$\rho_{Anom-xy} = R_{H-yx-0} B_z + R_s M(T, B)$.

Substituting Eq. (4.77) into Eq. (4.91), we obtain,

$$\rho_{ext-xy} \approx \frac{B_z}{nQ_e} + \frac{m}{n\tau Q_e^2}\left(\frac{S_{cx\pm}}{S_{cy\pm}}\right) - \frac{m}{n\tau Q_e^2}\left(\frac{j_x S_{cy\pm}}{nQ_e}\right) - \frac{S_{cy\pm}^2}{nQ_e} B_z. \quad (4.102)$$

The third term shows that Extended-Hall resistivity is dependent on the product of current and spin/magnetization, thus, contributes to $R_s M(T, B)$.

Remark: The third term, $\frac{m}{n\tau Q_e^2}\left(\frac{j_x S_{cy\pm}}{nQ_e}\right) = \rho_{Hall-xx}\left(\frac{j_x S_{cy\pm}}{nQ_e}\right)$, of Eq. (4.102) contributes to both spin Hall effect and Anomalous Hall effect. We show that, from the perspective of C-Spin-EM, GMR/TMR and family of Hall effects belong to the same category, i.e., they are, at least partially, caused by Spin-Lorentz-type force. C-Spin-EM indeed provides universal classical models for several fundamental quantum phenomena.

4.7.7. Temperature Dependence of Extended-Hall Effect

For describing the temperature-dependent behaviors of Extended-Hall effect, we utilize, for simplicity, a model of thermal velocity,

$$v_x = \frac{\sqrt{3kT}}{\sqrt{m}}, \quad (4.103)$$

and obtain temperature-dependent Extended-Hall coefficient/resistivity,

$$R_{ext-xx} = \frac{m}{n\tau Q_e^2 B_z} + \frac{\sqrt{m}}{nQ_e}\left(\frac{1}{\sqrt{3kT}}\right) S_{cy\pm}, \quad (4.104)$$

$$\rho_{ext-xx} = \frac{m}{n\tau Q_e^2} + \frac{\sqrt{m}}{nQ_e}\left(\frac{B_z}{\sqrt{3kT}}\right) S_{cy\pm}, \quad (4.90)$$

$$R_{ext-yx} = \frac{1}{nQ_e\left(1 - \frac{\sqrt{3kT}}{\sqrt{m}} S_{cx\pm}\right)}\left\{1 - \frac{S_{cx\pm}\sqrt{m}}{\sqrt{3kT}} - S_{cy\pm}\left(\frac{E_x}{B_z}\right)\right\}, \quad (4.105)$$

$$\rho_{ext-xy} = \frac{1}{nQ_e\left(1 - \frac{\sqrt{3kT}}{\sqrt{m}} S_{cx\pm}\right)}\left\{B_z - \sqrt{m} S_{cx\pm}\left(\frac{B_z}{\sqrt{3kT}}\right) - S_{cy\pm} E_x\right\}, \quad (4.106)$$

$$R_{ext-zx} = -\frac{(S_{cz\pm})}{nQ_e\left(1 - \frac{\sqrt{3kT}}{\sqrt{m}} S_{cx\pm}\right)}\left(\frac{E_x}{B_z}\right), \quad (4.107)$$

$$\rho_{ext-xz} = -\frac{(S_{cz\pm}) E_x}{nQ_e\left(1 - \frac{\sqrt{3kT}}{\sqrt{m}} S_{cx\pm}\right)}. \quad (4.108)$$

The second term of Eq. (4.107), $\sqrt{m}S_{cx\pm}\left(\frac{B_z}{\sqrt{3kT}}\right)$, shows that the Dual-Hall effect depends on the **B** field and temperature; The third term of Eq. (4.107), $S_{cy\pm}E_x$, shows temperature-independent.

4.7.8. Magnetic Aharonov-Casher-type Effect

Taking the cross and dot products of Spin-Lorentz-type force and spin of a test e-particle, respectively, we obtain

$$\mathbf{S_c}\times\mathbf{F_{SL}} = Q_e\mathbf{S_c}\times\mathbf{E_s} + Q_e\mathbf{S_c}\times(\mathbf{v}\times\mathbf{B_s}), \tag{4.110}$$

$$\mathbf{S_c}\cdot\mathbf{F_{SL}} = Q_e\mathbf{S_c}\cdot(\mathbf{v}\times\mathbf{B_s}) = \frac{Q_e}{m}(\mathbf{S_c}\times\mathbf{E})\cdot(\mathbf{S_c}\times\mathbf{p}). \tag{4.111}$$

When converting to quantum, the first term of Eq. (4.110) implies that spin-electric field $\mathbf{E_s}$ shifts the phase of a spinning e-particle.

Remark: (1) Denoted Eq. (4.110) as the Aharonov-Casher-type effect. (2) Eq. (4.111) is an Extended Spin-Orbit-Electric field Coupling.

4.7.9. More Effect

More effects of Spin-Lorentz-type force are

$$\mathbf{v}\times\mathbf{F_{SL}} = Q_e\mathbf{v}\times\mathbf{E_s} + Q_e\mathbf{v}\times(\mathbf{v}\times\mathbf{B_s}), \tag{4.112}$$

$$\mathbf{v}\cdot\mathbf{F_{SL}} = -Q_e\mathbf{v}\cdot(\mathbf{S_c}\times\mathbf{B}). \tag{4.113}$$

$$\mathbf{r}\times\mathbf{F_{SL}} = Q_e\mathbf{r}\times\mathbf{E_s} + Q_e\mathbf{r}\times(\mathbf{v}\times\mathbf{B_s}), \tag{4.114}$$

$$\mathbf{r}\cdot\mathbf{F_{SL}} = Q_e\mathbf{r}\cdot\mathbf{E_s} + \frac{Q_e}{m}\mathbf{E}\cdot(\mathbf{L}\times\mathbf{S_c}). \tag{4.115}$$

Remark: The second term, $\frac{Q_e}{m}\mathbf{E}\cdot(\mathbf{L}\times\mathbf{S_c})$, of Eq. (4.115) is Extended-Rashba SOC-1 of Eq. (4.57), which is derived from Hamiltonian. Namely the force, $\mathbf{F_{SL}}$, causes effect of Hamiltonian.

4.8. Lagrangian-Lorentz-type Force

Starting with Lagrangian's equation, $\frac{d}{dt}\frac{\partial\mathcal{L}_{total}}{\partial v} = \frac{\partial\mathcal{L}_{total}}{\partial r}$, where $\mathcal{L}_{total}$ is given by Eq. (4.48),

$$\mathcal{L}_{total} = \frac{1}{2}mv^2 + Q_e\mathbf{A}\cdot\mathbf{v} - Q_e\varphi + \frac{1}{2}a_{\mathcal{L}}(\mathbf{S_c})^2 + Q_e\mathbf{A_s}\cdot\mathbf{S_c} - Q_e\varphi_s + Q_e\mathbf{A_s}\cdot\mathbf{v} + Q_e\mathbf{A}\cdot\mathbf{S_c},$$

we derive a force, denoted it as the *Lagrangian-Lorentz-type Force* F_{LL},

$$\frac{1}{Q_e}F_{LL} = \mathbf{E} + \mathbf{v}\times\mathbf{B} + \mathbf{v}\times(\mathbf{S_c}\times\mathbf{E}) + \mathbf{S_c}\times(\mathbf{S_c}\times\mathbf{E}) + (\mathbf{S_c}\cdot\nabla)\mathbf{A_s} + (\mathbf{S_c}\cdot\nabla)\mathbf{A}. \tag{4.116}$$

The term, $\mathbf{E_s}$, and the term, $\mathbf{S_c}\times\mathbf{B}$, have canceled each other.

4.9. Spin-Potential-Coupling-Induced Force and Experiment

The Lagrangian-Lorentz-type force predicts a new force, Spin-Potential-Coupling-Induced force, represented by $(\mathbf{S_c}\cdot\nabla)\mathbf{A_s}$ and $(\mathbf{S_c}\cdot\nabla)\mathbf{A}$, which predict a new effect: a magnetic vector potential $\mathbf{A}$, even in an area where $\mathbf{B} = \mathbf{0}$, drives the originally static spinning e-charges to flow in opposite directions; the same to the spin-magnetic vector potential $\mathbf{A_s}$.

Testing Experiment 6: testing the Spin-potential Coupling. Let's consider a solenoid of p turns per unit length with the radius r, the current I flows in the direction of increasing ϕ, ρ is the distance from a point to the solenoid (Fig. 3). The magnetic vector potential is,

$A_{\phi} = \frac{\mu p I r^2}{2\rho}\hat{\phi}.$

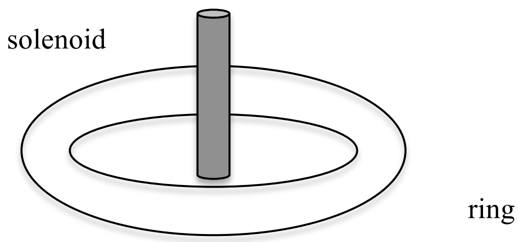

Fig. 3

The spin-potential-coupling-induced force on spinning e-particles is along the $\hat{\phi}$ direction,

$$\mathbf{F}_{\text{spin-poterntial}} = \frac{mv_{\phi}}{\tau} = Q_e(\mathbf{S_c} \cdot \nabla)\mathbf{A} = -Q_e \frac{\mu p I a^2}{2\rho^2} S_{c\rho\pm}\hat{\phi}. \quad (4.117)$$

Taking a ring and placing the "infinity long" solenoid through the center and perpendicular to the plane of the ring (Fig. 3), the "$S_{c\rho+}$ e-particles" and "$S_{c\rho-}$ e-particles" in the ring will go around the solenoid in the opposite directions to form opposite spin currents,

$$\mathbf{j}_{\text{spin-poterntial}} = nQ_e v_{\phi} = -\frac{nQ_e^2\tau}{m}\frac{\mu p I a^2}{2\rho^2} S_{c\rho\pm}\hat{\phi}. \quad (4.118)$$

For spinning e-particles, spin and e-charge are bonded together; spin current is accompanying with e-current. The $S_{c\rho+}$ e-particles and $S_{c\rho-}$ e-particles near the solenoid will flow fast than those farther from the solenoid.

For an open ring, e-particles with opposite spins will accumulate at opposite ends (Fig. 4). At equilibrium, the voltage between two ends is

$$V = -Q_e \frac{\pi \mu p I a^2}{\rho} S_{c\rho\pm}. \quad (4.119)$$

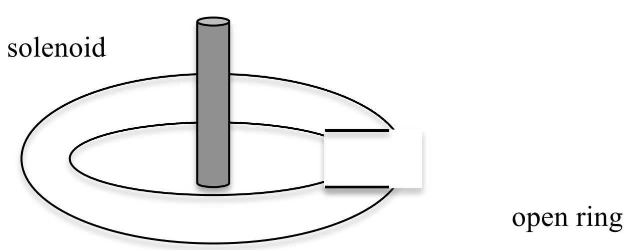

Fig. 4

Remark: the detection of this voltage/accumulation would prove the existence of the spin-potential-coupling-induced force and, thus, the Lagrangian-Lorentz-type force.

4.10. Effects of Lagrangian-Lorentz-type Force

4.10.1. Spin-Potential-Coupling Force Contributing to Aharonov–Bohm Effect

Let's consider a beam of spinning e-particles shooting at a solenoid. Experiencing the spin-potential-coupling force, the beam splits into opposite direction, due to the orientations of spins described by Eq. (4.113), and go around the solenoid, which contributes to Aharonov–Bohm Effect (Fig.5) [6].

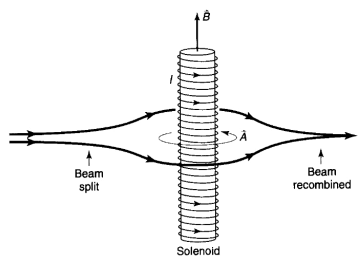

Fig. 5

Remark: Hamiltonian, Eq. (4.44), describes the A–B Effect. Now Lagrangian-Lorentz-type force contributes to the A-B effect, which implies that force is partial cause of this effect of Hamiltonian.

4.10.2. Lagrangian-Hall-type Effect/Topological Insulator

In regular Hall experiment, an external magnetic field **B** and an external electric field **E** are applied simultaneously. Let's study effects of the Lagrangian-Lorentz-type force. Considering a 3D material with width L in y-direction and thickness D in z direction, placed in a **B** field (z-axis) and an **E** field (x-axis), the longitudinal current density j_x along x-axis, same as Fig. 2.

For simplicity, ignoring the potential-dependent force, and let $S_{cx\pm} \equiv a$, $S_{cy\pm} \equiv b$, $S_{cz\pm} \equiv c$. At equilibrium, $v_y = v_z = 0$, the Lagrangian-Lorentz-type force gives the electric fields,

$$\frac{mv_x}{Q_e\tau} = E_x(1 - bb - cc),$$

$$E_x = \frac{mv_x}{Q_e\tau(1-bb-cc)}, \tag{4.120}$$

$$E_y = \frac{v_xB_z - bv_xE_x}{(1-av_x-aa-cc)}, \tag{4.121}$$

$$E_z = -\frac{cv_xE_x}{(1-av_x-aa-bb)}. \tag{4.122}$$

The Extended Hall parameters are,

(1) Extended Hall coefficients:

$$R_{E-H-xx} \equiv \frac{\mathbf{E}_x}{j_xB_z} = \frac{m}{n\tau Q_e^2 B_z(1-bb-cc)}, \tag{4.123}$$

$$R_{E-H-yx} \equiv \frac{\mathbf{E}_y}{j_xB_z} = \frac{B_z - bE_x}{nQ_eB_z(1-av_x-aa-cc)}, \tag{4.124}$$

$$R_{E-H-zx} \equiv \frac{\mathbf{E}_z}{j_xB_z} = -\frac{cE_x}{nQ_eB_z(1-av_x-aa-bb)}. \tag{4.125}$$

(2) Extended Hall conductivity,

$$\sigma_{SH-xx} \equiv \frac{j_x}{\mathbf{E}_x} = \frac{nQ_e^2\tau}{m} - \frac{nQ_e^2\tau}{m}S_{cy\pm}^2 - \frac{nQ_e^2\tau}{m}S_{cz\pm}^2, \tag{4.126}$$

$$\sigma_{E-H-xy} \equiv \frac{j_x}{\mathbf{E}_y} = \frac{nQ_e}{B_z-bE_x} - \frac{nQ_ev_x}{B_z-bE_x}S_{cx\pm} - \frac{nQ_e}{B_z-bE_x}S_{cx\pm}^2 - \frac{nQ_e}{B_z-bE_x}S_{cz\pm}^2, \tag{4.127}$$

$$\sigma_{E-H-xz} \equiv \frac{j_x}{\mathbf{E}_z} = -\frac{nQ_e}{cE_x} + \frac{nQ_ev_x}{cE_x}S_{cx\pm} + \frac{nQ_e}{cE_x}S_{cx\pm}^2 + \frac{nQ_e}{cE_x}S_{cy\pm}^2. \tag{4.128}$$

(3) Extended Hall Resistivity,

$$\rho_{ext-xx} \equiv R_{ext-xx}B_z = \frac{m}{n\tau Q_e^2(1-bb-cc)}, \tag{4.129}$$

$$\rho_{ext-xy} \equiv R_{ext-yx}B_z = \frac{B_z}{nQ_e(1-av_x-aa-cc)} - \frac{bE_x}{nQ_e(1-av_x-aa-cc)}, \tag{4.130}$$

$$\rho_{ext-xz} \equiv R_{ext-zx}B_z = -\frac{cE_x}{nQ_e(1-av_x-aa-bb)}. \quad (4.131)$$

We refer the effects described by Eq. (4.120) to Eq. (4.131) as the "Lagrangian-Hall-type Effect".

The Lagrangian-Lorentz-type Force induces Lagrangian-Hall-type effect, as Lorentz force induces Hall effect.

4.10.3. Spin-Larmor-type Precession

Lagrangian-Lorentz-type force $\mathbf{F_{LL}} = Q_e\mathbf{S_c}\times(\mathbf{S_c}\times\mathbf{E})$ induces Spin-Larmor-type precession,

$$\frac{d\mathbf{S_c}}{dt} = \mathbf{\Omega}\times\mathbf{S_c}, \quad (4.132)$$

$$\mathbf{\Omega} \equiv Q_e(\mathbf{S_c}\times\mathbf{E}). \quad (4.133)$$

Which is the same as the second term of Eq. (4.71).

4.10.4. Underlying Mechanism of Rashba Effect

The work done by Spin-magnetic field $\mathbf{B_s}$ is,

$$\mathbf{r}\cdot\mathbf{F_{LL}} = Q_e\mathbf{r}\cdot(\mathbf{v}\times\mathbf{B_s}) + Q_e\mathbf{r}\cdot(\mathbf{S_c}\times\mathbf{B_s}) = \frac{Q_e}{m}\mathbf{E}\cdot[\mathbf{L}\times\mathbf{S_c}] + \frac{Q_e}{m}\mathbf{E}\cdot[\boldsymbol{\ell}_s\times\mathbf{S_c}]. \quad (4.134)$$

where $\boldsymbol{\ell}_s \equiv m\mathbf{r}\times\mathbf{S_c}$. The first term of Eq. (4.134) is the Extended Rashba SOC-1 of Eq. (4.57).

The underlying mechanism of the Rashba effect is the spin-magnetic force.

4.10.5. Spin-magnetic-Rashba-type SOC

The Power of Lagrangian-Lorentz-type force acting on spin is,

$$\mathbf{v}\cdot\mathbf{F_{LL}} = Q_e\mathbf{v}\cdot(\mathbf{S_c}\times\mathbf{B_s}) = \frac{Q_e}{m}\mathbf{B_s}\cdot(\mathbf{p}\times\mathbf{S_c}). \quad (4.135)$$

We refer the term $\frac{Q_e}{m_e}\mathbf{B_s}\cdot(\mathbf{S_c}\times\mathbf{p})$ as the Spin-magnetic-Rashbe-type SOC.

4.11. Effects of Lorentz Force on Spin

4.11.1. Classical Origin of Aharonov–Casher effect

We suggest that Lorentz force acts on spin as,

$$\mathbf{S_c}\times\mathbf{F_L} = Q_e\mathbf{S_c}\times\mathbf{E} + \frac{Q_e}{m_e}\mathbf{S_c}\times(\mathbf{p}\times\mathbf{B}). \quad (4.136)$$

The first term provides a classical origin of the Aharonov–Casher effect.

Substituting the identity, $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$, into Eq. (4.136), we obtain

$$\mathbf{S_c}\times\mathbf{F_L} = Q_e\mathbf{S_c}\times\mathbf{E} + \frac{Q_e}{m_e}(\mathbf{S_c}\cdot\mathbf{B})\mathbf{p} - \frac{Q_e}{m_e}(\mathbf{S_c}\cdot\mathbf{p})\mathbf{B}. \quad (4.137)$$

Substituting the identity, $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\times\mathbf{C} + \mathbf{B}\times(\mathbf{A}\times\mathbf{C})$, into Eq. (4.136), we obtain,

$$\mathbf{S_c}\times\mathbf{F_L} = Q_e\mathbf{S_c}\times\mathbf{E} + \frac{Q_e}{m_e}(\mathbf{S_c}\times\mathbf{p})\times\mathbf{B} + \frac{Q_e}{m_e}\mathbf{p}\times(\mathbf{S_c}\times\mathbf{B}). \quad (4.138)$$

Substituting the definition of the $\mathbf{E_s}$ field into Eq. (4.138), we obtain,

$$\mathbf{S_c}\times\mathbf{F_L} = Q_e\mathbf{S_c}\times\mathbf{E} + \frac{Q_e}{m_e}(\mathbf{S_c}\times\mathbf{p})\times\mathbf{B} - \frac{Q_e}{m_e}\mathbf{p}\times\mathbf{E_s}. \quad (4.139)$$

4.11.2. Spin-Stark Effect; Magnetic-Rashba-type SOC

We can find how Lorentz force affects spin differently,

$$\mathbf{S_c}\cdot\mathbf{F_L} = Q_e\mathbf{S_c}\cdot\mathbf{E} + \frac{Q_e}{m_e}\mathbf{S_c}\cdot(\mathbf{p}\times\mathbf{B}) = Q_e\mathbf{S_c}\cdot\mathbf{E} + \frac{Q_e}{m_e}\mathbf{B}\cdot(\mathbf{S_c}\times\mathbf{p}). \quad (4.140)$$

The $Q_e \mathbf{S_c} \cdot \mathbf{E}$ causes spin-Stark effect. The $\frac{Q_e}{m_e} \mathbf{B} \cdot (\mathbf{S_c} \times \mathbf{p})$ is a magnetic-Rashba-type SOC.

5. Summary and Discussion

Euclidean geometry is the first self-consistent mathematical systems established based on few axioms, and deriving other theorems from axioms.

Based on mathematical vector identities, we establish self-consistent UMFE that universally describes classical physical fields induced respectively by velocity and classical spin of a source.

Combining UMFE and Coulomb's law, we derived Extended EM, which: (1) justifies UMFE; (2) shows that the experiments-based EM have their mathematical origin; and (3) the uniform motion of an e-particle inevitably induce an axial vector magnetic field, which is predetermined mathematically. Moreover Extended EM predicts new effects that the spatially varying velocity of e-particle induces axial magnetic field and axial induced electric field.

Combining UMFE and Coulomb's law, we derived C-Spin-EM in the perspective of fundamental physics.

C-Spin-EM is powerful and fruitful, and achieves the following:

(1) Derives spin wave.
(2) Derives Spin-Lorentz-type force and Lagrangian-Lorentz-type force, which, for 3D model, cause Dual-Hall Effect, Extended-Hall effect, Lagrangian-Hall-type Effect, and Temperature Dependence of Extended-Hall effect; also cause Extended Rashba SOC-1, -2, -3.
(3) Extended-Hall effect contributes universally to zero longitudinal Hall coefficient/resistivity, GMR/TMR, Anomalous Hall effect, Spin Hall effect, and topological insulator.
(4) Predicts Spin-Potential-Coupling-Induces force that contributes to Aharonov–Bohm Effect.
(5) Provides classical counterparts of Aharonov-Bohm effect; Aharonov-Casher effect; Stark effect; Larmor Precession; Zeeman effect.
(6) Proposes several experiments to test proposed effects, such as, definition of spin-electric and spin-magnetic fields, Spin-Aharonov–Bohm effect, Dual-Hall Effect/Topological Insulator, whether $\rho_{ext-xx}(\mathbf{B}) \approx 0$ of GMR/TMR, Spin-Potential-Coupling-Induced force, Zeeman effect/Extended-Rashba SOC.

We argue that the self-consistency, powerfulness and fruitfulness are evidences supporting C-Spin-EM.

The mathematical identities lead to the physical dualities. UMFE provides mathematical origins of physical dualities between Extended EM and C-Spin-EM.

The Lagrangian-Lorentz-type force and the Hamiltonian, Eq. (4.50), are derived from the same Lagrangian. We suggest that there is a duality between the Lagrangian-Lorentz-type force and the Hamiltonian, i.e., an effect of Hamiltonian corresponds to an effect of Lagrangian-Lorentz-type force, and vice versa, which is heuristic for exploring new effects.

The extent of validity of C-Spin-EM is its extent to correctly predict and agree with experimental results.

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Appendix : Inver-square Law for Spin

Postulate a physically hypothetical $Q_{spin-momopole}$ (abbreviated Q_{s-m}), call "spin-charge", which is a spin counterpart of e-particle, and satisfies an inverse-square law,

$$\nabla \cdot \mathbf{E_s} = Q_{s-m}.$$