Extended Hubble Law for the Accelerating and Jerking Universe

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Abstract

In the 1998 and the 2016, scientists report the accelerating universe and the jerking universe, respectively. Hubble law was derived based on the uniformly expanding universe and may be extended to fit the accelerating and jerking universe. For this aim, we generalize the linear distance-velocity and distance-redshift relations of Hubble law to a non-linear distance-movement and a distance-redshift-movement relation of the generalized Hubble law. We suggest that the generalized Hubble law are worth to pursue and may be tested by introducing it into cosmological study.

Key words: distance-velocity relation, distance-redshift relation, Hubble law, accelerating universe, redshift, Cosmology.

1. Expansion of Universe

In the 1929, Hubble found that the universe is expanding [1], $\dot{a} > 0$. In the 1998, Scientists reported the observations that the expansion of the universe is accelerating [2], $\ddot{a} > 0$, and the acceleration is assumed to be uniform, $\dddot{a} = 0$. In the 2016 [3], Scientists report that even that acceleration is faster than expected. If the 2016 observation is confirmed, which could mean that the acceleration is accelerating, $\dddot{a} > 0$, there is a discrepancy in Hubble constant obtained from the 2016 observation and from the 2013 data of European Planck spacecraft. Following classical mechanics, call $\dddot{a}(t)$ the "jerk". There is no evidence showing whether the jerk is uniform or not. Historically, one has experienced a chain of transitions of concepts of the evolution of the universe:

$$\boxed{\text{Static} \rightarrow \text{uniformly expansion} \rightarrow \text{accelerating expansion} \rightarrow \text{jerking expansion}}$$

Hubble started observationally the first transition by finding the linear distance-redshift and distance-velocity relations between extra-galactic nebulae, both known as Hubble's law [1]. At Hubble's time, the receding velocity was considered as constant, no acceleration involved in Hubble law. The application of Hubble law should be limited principally to the uniformly expanding universe. The 1998 observation starts the second transition and is explained by dark energy model. In this model, Hubble law plays the fundamental role, which implies that acceleration has no effect on Hubble-parameter-related cosmological phenomena. Obviously Hubble law is over simplified for the accelerating universe. The 2016 observation starts the third transition. Indeed the linear Hubble law is not applicable for the jerking universe as well.

Guided by observations of acceleration and jerk, we suggest that the distance-velocity and the distance-redshift relations may be generalized to distance-velocity-acceleration-jerk and distance-redshift-acceleration-jerk relations, respectively, to describe the accelerating and jerking universe.

2. Hubble Law: Distance-Velocity Relation for Uniformly Expanding Universe

Both Newton and Einstein theories of gravity are second order derivative and cannot describe either jerk or higher order derivatives phenomena [4]. For deriving an extended Hubble law, we apply a mathematical approach. Let's start with Taylor series of a radial physical distance between nebulae,

$$r(t_1) = r(t) + \dot{r}(t)(t_1 - t) + \frac{1}{2!}\ddot{r}(t)(t_1 - t)^2 + \frac{1}{3!}\dddot{r}(t)(t_1 - t)^3 + \cdots. \qquad (1)$$

We notice a surprising fact that the history of observationally discovering the expansion of the universe precisely matches Eq. (1), one term at time (shown below).

Static ---1929	Uniformly expansion: 1929	Accelerating expansion: 1998	Jerking expansion: 2016	Jouncing Expansion: ?
$r(t)$	$\dot{r}(t)$	$\ddot{r}(t)$	$\dddot{r}$	$r^{(4)}$

Superscript "(4)" denotes the 4th order derivative with respect to time. Call "$r^{(4)}$" "Jounce". This fact is heuristic and guides for deeply understanding and describing the expanding universe. We believe there should be only one dynamics mechanism for driving acceleration, jerk, and all higher order derivatives [4]. The mechanism is out of scope of this article.

Before Friedmann, Lemaitre and Hubble, it was believed that the universe was static, i.e., the physical distance between nebulae didn't change with time,

$r(t_1) = r(t)$ and $\dot{r}(t) = 0$,

which corresponds to the first term of right hand side of Eq. (1). The concept of the static universe leaded Einstein to introduce the cosmological constant to balance the attractive gravitational force. The first term, $r(t)$, is a constant at an initial time t. When study the expansion of the universe, we are interested in the change of distance and, thus absorb it into the left hand side of Eq. (1). Then Eq. (1) becomes,

$$r(t_1) = \dot{r}(t)(t_1 - t) + \frac{1}{2!}\ddot{r}(t)(t_1 - t)^2 + \frac{1}{3!}\dddot{r}(t)(t_1 - t)^3 + \cdots. \qquad (2)$$

In the 1929, Hubble established the distance-velocity relation,

$$\dot{r} = Hr. \qquad (3)$$

The farther an object is away, the faster it is receding. Hubble law was established based on the observation of the movement without observing acceleration. Hubble realized this fact and named it as the "distance-velocity relation". This name reminds one that, precisely speaking, Hubble law is applicable for uniformly expanding universe only.

For a clear picture, rewrite the distance-velocity relation of Hubble law, Eq. (3), as,

$$r(t_1) = \dot{r}(t)\left(\frac{1}{H}\right). \qquad (4)$$

The faster an object is uniformly receding, the farther it is away. $\frac{1}{H}$ is the Hubble time. In this form, the physical distance corresponds to the first term of right hand side of Eq. (2). Comparing Hubble law, Eq. (4), with Eq. (2), we obtain,

$$\frac{1}{H} = (t_1 - t). \qquad (5)$$

Both Eq. (4) and Eq. (5) define the Hubble parameter. For uniform motion, both definitions are equivalent. However, for non-uniform motion, they are different. We will show that the definition Eq. (4) is modified by acceleration and jerk.

3. Distance-Velocity-Acceleration Relation for Uniformly Accelerating Universe

In the 1998, Perlmutter, Schmidt, and Riess reported that the expansion of the universe is accelerating, and it was assumed that the acceleration is uniform,

$$\ddot{r}(t_1) = \ddot{r}(t) \qquad \text{or} \qquad \dddot{r}(t_1) = 0. \qquad (6)$$

The 1998 observation corresponds to the second term of right hand side of Taylor series, Eq. (2). Therefore the distance-velocity relation of Hubble law needs to be extended to a "distance-velocity-acceleration relation". In classical mechanics, for uniformly accelerating objects, the distance-velocity-acceleration relation is,

$$r(t_1) = \dot{r}(t)(t_1 - t) + \frac{1}{2}\ddot{r}(t)(t_1 - t)^2. \qquad (7)$$

Here $\dot{r}(t)$ is the receding velocity at time t prior to t_1, and increases continuously.

The combination of the velocity and acceleration determines the form of an extended Hubble law. Let's define a distance-velocity-acceleration relation as the extended distance-velocity relation,

$$r(t_1) \equiv \alpha\dot{r}(t) + \beta\ddot{r}(t). \qquad (8)$$

Comparing Eq. (4, 5, 7 and 8), we assume,

$$\alpha \equiv \frac{1}{H}, \qquad \text{and} \qquad \beta \equiv \frac{1}{2H^2}, \qquad (9)$$

and obtain the extended Hubble law for the "accelerating universe", as,

$$r(t_1) = \dot{r}(t)\left(\frac{1}{H}\right) + \frac{1}{2}\ddot{r}(t)\left(\frac{1}{H^2}\right), \qquad (10)$$

which corresponds to the first two terms of right hand side of Taylor series, Eq. (2). The faster an object is originally receding and uniformly accelerating, the farther it is away.

Now let's convert parameter at time t to that at time t_1. We know

$$\dot{r}(t_1) = \dot{r}(t) + \ddot{r}(t)(t_1 - t). \qquad (11)$$

Substituting Eq. (6 and 11) into Eq. (10), we re-express the extended Hubble law as,

$$r(t_1) = \dot{r}(t_1)\left(\frac{1}{H}\right) - \frac{1}{2}\ddot{r}(t_1)\frac{1}{H^2}. \qquad (12)$$

Eq. (12) implies that the distance is equal to its present velocity times the time interval, subtracts the distance due to the acceleration, and indicates that an accelerating universe is older than a uniformly expanding universe of same size and same present velocity.

The extended Hubble law may be expressed in different forms as the following,

$$\frac{\dot{r}(t_1)}{r(t_1)} = H\left(1 + \frac{1}{2}\frac{\ddot{r}}{r}\frac{1}{H^2}\right), \tag{13}$$

$$H = \frac{1}{2}\frac{\dot{r}}{r}\left\{1 + \sqrt{1 - 2\frac{\ddot{r}(t_1)r(t_1)}{\dot{r}^2}}\right\}. \tag{14}$$

Note with acceleration, $H \neq \frac{\dot{r}}{r}$, i.e., the definition of Hubble parameter, Eq. (4), is no longer valid. Eq. (14) is the extension of Eq. (4) and may be considered as a new definition of Hubble parameter in terms of velocity and acceleration. For small acceleration, Eq. (14) becomes

$$H \approx \frac{\dot{r}}{r} - \frac{1}{2}\frac{\ddot{r}}{\dot{r}} \tag{15}$$

When ignoring the acceleration, Eq. (12-15) reduces to the convention Hubble law.

4. Distance-Velocity-Acceleration-Jerk Relation for "Jerking Universe"

In the 2016, Riess et al, report that the acceleration of the expansion of the universe is accelerating, denote as "jerking Universe", which corresponds to the third term of right hand side of Taylor series, Eq. (2). Empirically, to detect the higher order derivative terms of Eq. (2) requires more sensitive tool. Theoretically, Newton and Einstein's theories of gravity don't have a mechanism to drive jerk and higher order derivative terms unless we extend theories of gravity [4]. Now the jerk is assumed to be constant,

$$\dddot{r}(t_1) = \dddot{r}(t) \qquad \text{and} \qquad r^{(4)}(t_1) = 0. \tag{16}$$

Classical mechanics gives the distance,

$$r(t_1) = \dot{r}(t)(t_1 - t) + \frac{1}{2}\ddot{r}(t)(t_1 - t)^2 + \frac{1}{3!}\dddot{r}(t)(t_1 - t)^3. \tag{17}$$

Not only velocity and acceleration but also jerk of objects determines the form of distance. As the consequence, we propose a distance-velocity-acceleration-jerk relation, as,

$$r(t_1) = \alpha\dot{r}(t) + \beta\ddot{r}(t) + \gamma\dddot{r}(t), \tag{18}$$

Comparing Eq. (8, 9, 17 and 18), we assume,

$$\alpha \equiv \frac{1}{H}, \qquad \beta \equiv \frac{1}{2!H^2}, \qquad \gamma \equiv \frac{1}{3!H^3}. \tag{19}$$

Substituting Eq. (19) into Eq. (18), the extended Hubble law may be written as,

$$r(t_1) = \dot{r}(t)\left(\frac{1}{H}\right) + \ddot{r}(t)\left(\frac{1}{2!H^2}\right) + \dddot{r}(t)\left(\frac{1}{3!H^3}\right), \tag{20}$$

which agrees with the first three terms of Taylor series, Eq. (2). The faster an object is originally receding, accelerating, and uniformly jerking, the farther it is away.

Now let's express the parameters of Eq. (20) at time t_1. We have,

$$\ddot{r}(t_1) = \ddot{r}(t) + \dddot{r}(t)(t_1 - t), \tag{21}$$

$$\dot{r}(t_1) = \dot{r}(t) + \ddot{r}(t)(t_1 - t) + \frac{1}{2}\dddot{r}(t)(t_1 - t)^2, \tag{22}$$

The right hand side of Eq. (21 and 22) implies that the acceleration and velocity at a prior time t are slower than that at later time t_1. Substituting Eq. (21 and 22) into Eq. (20), we obtain the distant-velocity-acceleration-jerk relation of the extended Hubble law in different forms as the following,

$$r(t_1) = \frac{\dot{r}(t_1)}{H} - \frac{\ddot{r}(t_1)}{2H^2} + \frac{\dddot{r}(t_1)}{3!H^3}, \tag{23}$$

$$\frac{\dot{r}(t_1)}{r(t_1)} = H\left\{1 - \frac{1}{2}q_2 + \frac{1}{3!}q_3\right\}, \tag{24}$$

where

$$q_n \equiv -\frac{r^{(n)}(t_1)}{r(t_1)H^n}. \tag{25}$$

Generally, q_2 and q_3 are the "deceleration-judging parameter" and "jerk-judging parameter", respectively. Without acceleration and jerk, Eq. (23 and 24) reduces to the linear Hubble law.

5. General Distance-Movement Relation

The most general distance-movement relation has the following form,

$$r(t_1) = \alpha\dot{r}(t) + \beta\ddot{r}(t) + \gamma\dddot{r}(t) + \cdots + \delta r^{(n)} + \cdots, \tag{26}$$

$$\alpha \equiv \frac{1}{H}, \qquad \beta \equiv \frac{1}{2!H^2}, \qquad \gamma \equiv \frac{1}{3!H^3}, \qquad \delta = \frac{1}{n!H^n}. \tag{27}$$

The superscript "(n)" denotes the n^{th} order derivative with respect to time. The general distance-movement relation is equivalent to Taylor series of physical distance. Expressing parameters of Eq. (26) at time t_1, we obtain the generalized Hubble law,

$$\frac{\dot{r}(t_1)}{r(t_1)} = H\left\{1 + \sum_{n=2}^{\infty}\frac{(-1)^n}{n!}\frac{r^{(n)}(t_1)}{r(t_1)H^n}\right\} = H\left\{1 + \sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n!}q_n\right\}. \tag{28}$$

With an appropriate dynamical mechanism, the general Hubble law predicts that the expansion of the universe is jouncing, i.e., the jerking is accelerating.

As an application of Eq. (28), Friedmann equation becomes,

$$\left(\frac{\dot{a}(t_1)}{a(t_1)}\right)^2 = H^2\left(1+\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n!}q_n\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{r^2} + \frac{\Lambda c^2}{3}. \quad (29)$$

Hubble law is the first approximation of the extended Hubble law.

6. Distance-Redshift-Acceleration-Jerk Relation

First let's consider two wave crests. One obtains the relation,

$$\frac{\lambda_0}{\lambda_1} = \frac{r(t_0)}{r(t_1)}, \quad (30)$$

where λ_0 and λ_1 is the wavelength at observing and emitting time, respectively; $r(t_0)$ and $r(t_1)$ is the scale factor at observing and emitting time, respectively. The universe is expanding with jerk as observed in the 2016 observation; it is possible to expand at even higher order derivative. To study the effects of acceleration and jerk on the distance-redshift relation, we need to introduce them into the expression of either $r(t_0)$ or $r(t_1)$. Following the same method of extending the distance-velocity relation, we apply Taylor series again

$$r(t_1) = r(t_0) + \dot{r}(t_0)(t_1 - t_0) + \frac{\ddot{r}(t_0)}{2}(t_1 - t_0)^2 + \frac{\dddot{r}(t_0)}{3!}(t_1 - t_0)^3 + \cdots. \quad (31)$$

Based on Eq. (31), we define the general distance-redshift-motion relation as,

$$\frac{r(t_0)}{r(t_1)} = \frac{1}{1-\frac{\dot{r}(t_0)}{r(t_0)}(t_0-t_1)+\frac{\ddot{r}(t_0)}{2r(t_0)}(t_0-t_1)^2-\frac{\dddot{r}(t_0)}{3!r(t_0)}(t_0-t_1)^3+\cdots}. \quad (32)$$

For nearby sources, Eq. (32) becomes,

$$\frac{r(t_0)}{r(t_1)} \approx 1 + \frac{\dot{r}(t_0)}{r(t_0)}(t_0 - t_1) - \frac{\ddot{r}(t_0)}{2r(t_0)}(t_0 - t_1)^2 + \frac{\dddot{r}(t_0)}{3!r(t_0)}(t_0 - t_1)^3 - \cdots. \quad (33)$$

Combining Eq. (30) and Eq. (33), we obtain the general distance-redshift-motion relation,

$$1 + Z = 1 + \frac{\dot{r}(t_0)}{r(t_0)}(t_0 - t_1) - \frac{\ddot{r}(t_0)}{2r(t_0)}(t_0 - t_1)^2 + \frac{\dddot{r}(t_0)}{3!r(t_0)}(t_0 - t_1)^3 - \cdots. \quad (34)$$

Therefore, we obtain redshift Z. In terms of Hubble constant and the rate of changing of Hubble redshift Z_{H0}, Z is

$$Z \approx Z_{H0} - \frac{\dot{Z}_{H0}}{2H_0} + \frac{\ddot{Z}_{H0}}{3!H_0^2} - \frac{\dddot{Z}_{H0}}{4!H_0^3} + \cdots, \qquad Z_{H0} = \frac{H_0 r(t_0)}{c}. \tag{35}$$

The redshift of an object drifts due to its acceleration, $\dot{Z}_{H0}$. The jerk causes accelerating drift, $\ddot{Z}_{H0}$.

Or, in terms of motion and Hubble redshift, Z is

$$Z \approx Z_{H0} - \frac{\ddot{r}(t_0)r(t_0)}{2\dot{r}^2(t_0)} Z_{H0}^2 + \frac{\dddot{r}(t_0)r^2(t_0)}{3!\dot{r}^3(t_0)} Z_{H0}^3 - \frac{r^{(4)}(t_0)r^3(t_0)}{4!\dot{r}^4(t_0)} Z_{H0}^4 + \cdots. \tag{36}$$

Let's introduce a "redshift-judging parameter" q_{R0}^n as,

$$q_{R0}^n \equiv -\frac{r^{(n)}(t_0)r^{n-1}(t_0)}{\dot{r}^n(t_0)}. \tag{37}$$

Substituting q_{R0}^n into Eq. (36), we obtain,

$$Z = Z_{H0} + \frac{1}{2} q_{R0}^2 Z_{H0}^2 - \frac{1}{3!} q_{R0}^3 Z_{H0}^3 + \frac{1}{4!} q_{R0}^4 Z_{H0}^4 - \cdots. \tag{38}$$

7. Summary

We have generalized the distance-velocity and distance-redshift relations of Hubble law to the distance-movement and distance-redshift-movement relations of the general Hubble law. The table shows that the history of discovering the expansion of the universe proves, one term at time, the capability of Taylor series, Eq. (2 and 30), in describing the relations between distance, movements, and redshift.

	Distance-Movement Relation	Distance-Redshift-Momentum Relation: $1 + Z = \frac{r(t_0)}{r(t)}$
Expanding Universe	$r = \frac{\dot{r}}{H}$	$Z = Z_{H0} = \frac{H_0 r(t_0)}{c}$
Accelerating Universe	$r = \frac{\dot{r}}{H} - \frac{\ddot{r}}{2H^2}$	$Z = Z_{H0} - \frac{\ddot{r}(t_0)r(t_0)}{2\dot{r}^2(t_0)} Z_{H0}^2$
Jerking Universe	$r = \frac{\dot{r}}{H} - \frac{\ddot{r}}{2H^2} + \frac{\dddot{r}}{3!\,H^3}$	$Z = Z_{H0} - \frac{\ddot{r}(t_0)r(t_0)}{2\dot{r}^2(t_0)} Z_{H0}^2 + \frac{\dddot{r}(t_0)r^2(t_0)}{3!\,\dot{r}^3(t_0)} Z_{H0}^3$
Jouncing Universe	$r = \frac{\dot{r}}{H} - \frac{\ddot{r}}{2H^2} + \frac{\dddot{r}}{3!\,H^3} - \frac{r^{(4)}}{4!\,H^4}$	$Z = Z_{H0} - \frac{\dot{Z}_{H0}}{2H_0} + \frac{\ddot{Z}_{H0}}{3!\,H_0^2} - \frac{\dddot{Z}_{H0}}{4!\,H_0^3}$

We suggest that the distance-movement and distance-redshift-movement relations of the generalized Hubble law are worth to pursue and may be tested by introducing those relations into cosmological study. The expansion of the universe may be an evidence of the generalized Hubble law.

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